BINARY STARS - DETERMINING MASSES AND THE MASS FUNCTION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 18 Jun 2023.

In a binary star system, we can get estimates of the masses of the two stars given enough measurements of their motions in their orbits.

Visual binaries. First, suppose the binary is a visual binary, that is, both stars are visible directly. This will occur if their separation is large enough and both components are bright enough relative to each other. Now suppose that the orbital plane is perpendicular to the line of sight, and that we observe the binary star through at least one of its periods of revolution. We can then measure the angle $\alpha_i,\ i=1,2$ subtended by the semimajor axis a_i of the orbit of each star. Then, from the centre of mass condition we can get the mass ratio:

$$m_1 r_1 = m_2 r_2 \tag{1}$$

$$\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1} \tag{2}$$

Note that we don't need to know the distance to the stars to get their mass ratio.

To find the individual masses, we need another condition, which can be found in Kepler's third law:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3 \tag{3}$$

where P is the period and a is the semimajor axis of the orbit of the reduced mass: $a = a_1 + a_2$. This time we need to know the actual length of a rather than just its ratio to something else, so we do need the distance to the stars. The distance can be obtained by some other method, such as parallax.

However, the above analysis assumed that the orbital plane was perpendicular to the line of sight, that is, that its angle of inclination $i=0$. In practice, this is rarely the case so we need to adjust the formulas to include this effect. For the special case (yes, another one) where the semiminor axis lies in the plane perpendicular to the line of sight, but the orbital plane is rotated about the semiminor axis by the angle i, the observed angles subtended by the semimajor axes of the two stars are both shortened by a factor of $\cos i$ (see Carroll & Ostlie's Fig. 7.4), so from 2, the mass ratio can still be obtained by just taking the ratio of the observed angles, since the common factor of $\cos i$ cancels out. However, even if we know the distance d to the stars, we must also know i to use Kepler's third law to get the sum of masses. Suppose $\tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2$ is the observed angle subtended by the reduced mass's semimajor axis. Then the actual semimajor axis is

$$a = \frac{d\tilde{\alpha}}{\cos i} \tag{4}$$

so

$$m_1 + m_2 = \frac{4\pi^2}{GP^2}\left(\frac{d\tilde{\alpha}}{\cos i}\right)^3 \tag{5}$$

We can estimate this by observing that if we project an ellipse tilted by an angle i onto the observing plane, we get another ellipse with a different eccentricity (for example, if you tilt an ellipse at just the right angle, the projection will be a circle). More importantly, the projection of the foci will *not* coincide with the foci of the projected ellipse. In physical terms, this means that the projection of the true centre of mass won't coincide with a focus of the projected ellipse, and this discrepancy can be detected by careful observation. The problem then is to figure out the actual ellipse and its inclination angle that gives the observed projection. Carroll & Ostlie leave the discussion at this point so we won't pursue it here.

Spectroscopic binaries. If the two stars are too close together to be resolved visually, they can still be detected spectroscopically. If the inclination angle is greater than zero, some component of the stars' orbital velocity is along the line of sight, so the spectral lines of the two stars are Doppler shifted relative to each other. If the actual velocity of star j is v_j, the radial component is

$$v_{j,r} = v_j \sin i \tag{6}$$

If the orbits are circular, then the velocity of each star is constant throughout its orbit and is

$$v_j = \frac{2\pi r_j}{P} \tag{7}$$

so from 2 we can get the mass ratio from the radial velocity measurements:

$$\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{v_2}{v_1} = \frac{v_{2,r}}{v_{1,r}} \tag{8}$$

As before, we need to know i to use Kepler's third law to get the sum of the masses. In the circular orbit case, the semimajor axis a is just the radius of the orbit, so

$$a = a_1 + a_2 = \frac{P}{2\pi}(v_1 + v_2) \tag{9}$$

From 3, we get

$$m_1 + m_2 = \frac{P}{2\pi G}(v_1 + v_2)^3 \tag{10}$$

$$= \frac{P}{2\pi G}\left(\frac{v_{1,r} + v_{2,r}}{\sin i}\right)^3 \tag{11}$$

Note that we don't need the distance d to the star system, but we do need to be able to measure the radial velocities of both stars. This is possible unless one component is much brighter than the other, in which case only one set of spectral lines is visible. In this case, we have

$$v_{2,r} = v_{1,r}\frac{m_1}{m_2} \tag{12}$$

$$\frac{m_2^3}{(m_1 + m_2)^2}\sin^3 i = \frac{P}{2\pi G}v_{1,r}^3 \tag{13}$$

The last line places all the observable quantities on the RHS, with the unknowns on the left. The quantity of the RHS is called the *mass function*. Since $\sin i \le 1$ and $\frac{m_2^2}{(m_1+m_2)^2} = \frac{1}{(1+m_1/m_2)^2} < 1$, the LHS is always less than m_2 so the mass function gives a lower bound on the mass of the unseen star. If this lower bound is high enough, it can give strong evidence that the unseen companion is a black hole.

In the case where both stars are visible, so that both $v_{1,r}$ and $v_{2,r}$ can be measured, we can use 8 and 13 to find the actual masses if we know i. However, i is usually unknown, so what is often done is to use an average value $\langle \sin^3 i \rangle$ so that the two equations can be solved for the masses. As we'll see, there is a strong correlation between the luminosities and masses of stars, so if we classify stars according to their luminosities (and surface temperatures), we can then use these two equations to get an estimate of the mass of stars in that class. When luminosity is plotted against mass on a log-log graph, the result is a straight line.

In order to calculate $\langle \sin^3 i \rangle$, we need to take into account that it is more likely that we will detect a spectroscopic binary if i is larger, since for i near zero, the orbital plane is essentially the observation plane and the radial velocities will be very small. To get an average value of $\langle \sin^3 i \rangle$, we therefore should use a weighting function that emphasizes values of i closer to 90°. In principle, we could use any monotonically increasing weighting function $w(i)$ in the range $0 \le i \le \frac{\pi}{2}$, but for the sake of argument, we can use $w(i) = \sin i$. This is normalized since

$$\int_0^{\pi/2} w(i)\, di = 1 \tag{14}$$

The average then becomes

$$\left\langle \sin^3 i \right\rangle = \int_0^{\pi/2} w(i) \sin^3 i\; di \tag{15}$$

$$\int_0^{\pi/2} (\sin i)\left(\sin^3 i\right) di \tag{16}$$

$$= \left[\frac{3i}{8} - \cos i \left(\frac{3}{8}\sin i + \frac{1}{4}\sin^3 i\right)\right]_0^{\pi/2} \tag{17}$$

$$= \frac{3\pi}{16} \tag{18}$$

$$\approx 0.589 \tag{19}$$

Another possible weighting function is just the (normalized) straight line

$$w(i) = \frac{8}{\pi^2} i \tag{20}$$

$$\left\langle \sin^3 i \right\rangle = \frac{8}{\pi^2}\int_0^{\pi/2} i\left(\sin^3 i\right) di \tag{21}$$

$$= \frac{56}{9\pi^2} \tag{22}$$

$$\approx 0.63 \tag{23}$$

Integrals were done using Maple.

PINGBACKS

Pingback: Eclipsing binary stars
Pingback: Sirius binary star system
Pingback: Spectroscopic binary stars - Zeta Phe
Pingback: Spectroscopic, eclipsing binary stars - mass, radius and temperature
Pingback: Extrasolar planets - some detection methods
Pingback: Radial velocities of a binary system - computer model