ELLIPSES

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In a two-body solar system (that is, where we have only the Sun and a single planet or other object orbiting it), it turns out that the planet's orbit is a conic section (ellipse, parabola or hyperbola) with the centre of mass of the system at one of the foci (or the single focus, in the case of a parabola) of the orbit. For any bound orbit (that is, one in which the planet never escapes to infinity), the orbital curve is an ellipse, so it's useful to review a few properties of ellipses.

To define an ellipse, first specify two points which serve as the foci (see Fig. 2.4 in Carroll & Ostlie). The ellipse is then the set of points such that the sum of the distances of a given point from the two foci is a constant, namely $2a$, where a is called the *semimajor axis* (and $2a$ is the *major axis*). If the distance from a point P on the ellipse to one focus is r' and to the other focus is r, then

$$r' + r = 2a \tag{1}$$

If we place the ellipse in a rectangular coordinate system with the line between the foci on the x axis and the origin at the centre of the ellipse, then the distance along the y axis from the origin to the ellipse is known as b, the *semiminor axis*. The *eccentricity* e of the ellipse is defined as the distance between the foci divided by the major axis, so that the distance between the foci is $2ae$. If the two foci coincide, the distance between them is zero so $e = 0$, $r' = r = a$ and we get a circle. If the foci coincide with the ends of the major axis, then $ae = a$ and $e = 1$, giving just a line segment along the x axis extending from $x = -a$ to $x = +a$. In the general case, if we draw a right angled triangle with vertices at one of the foci, the centre, and the point $(0,b)$, then the hypotenuse of this triangle is a so by Pythagoras we have

$$a^2e^2 + b^2 = a^2 \tag{2}$$
$$b^2 = a^2\left(1 - e^2\right) \tag{3}$$
$$e^2 = 1 - \frac{b^2}{a^2} \tag{4}$$

If we use a polar coordinate system with the origin at the right-hand focus, then the angle θ between the x axis and the line r from that focus to the ellipse is the polar angle. To get the equation of an ellipse in polar coordinates, we can consider the triangle with its vertices at the polar origin, the point on the ellipse with coordinates (r,θ) and the left-hand focus. The triangle's angle at the origin is then $\pi - \theta$ so by the law of cosines we have

$$\left(r'\right)^2 = 4a^2e^2 + r^2 - 4rae\cos(\pi - \theta) \tag{5}$$
$$= 4a^2e^2 + r^2 + 4rae\cos\theta \tag{6}$$

From 1 we have

$$\left(r'\right)^2 = 4a^2 + r^2 - 4ar \tag{7}$$

so

$$4a^2 - 4ar = 4a^2e^2 + 4rae\cos\theta \tag{8}$$
$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta} \tag{9}$$
$$= \frac{b^2}{a(1 + e\cos\theta)} \tag{10}$$

We can derive a rectangular coordinate version of the equation for an ellipse by two methods. We can start with 9 and shift the origin to the centre of the ellipse, so that a point on the ellipse has rectangular coordinates

$$x = ae + r\cos\theta \tag{11}$$
$$y = r\sin\theta \tag{12}$$

Now (since we know the answer we're looking to prove) we can form the following quantity, using 4 and 10:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{1}{a^2}\left(a\sqrt{1-\frac{b^2}{a^2}}+r\cos\theta\right)^2+\frac{r^2\sin^2\theta}{b^2} \tag{13}$$

$$= \frac{1}{a^2}\left(a\sqrt{1-\frac{b^2}{a^2}}+\frac{b^2\cos\theta}{a\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)}\right)^2+\frac{b^2\sin^2\theta}{a^2\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)^2} \tag{14}$$

$$= \frac{1}{a^4\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)^2}\times$$
$$\left[\left(a^2\sqrt{1-\frac{b^2}{a^2}}\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)+b^2\cos\theta\right)^2+a^2b^2\sin^2\theta\right] \tag{15}$$

$$= \frac{a^4}{a^4\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)^2}\left[\left(1-\frac{b^2}{a^2}\right)\cos^2\theta+2\sqrt{1-\frac{b^2}{a^2}}\cos\theta+1\right] \tag{16}$$

$$= \frac{\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)^2}{\left(1+\sqrt{1-\frac{b^2}{a^2}}\cos\theta\right)^2} \tag{17}$$

$$= 1 \tag{18}$$

The other method is to use the definition 1 and Pythagoras again. We have

$$\left(r'\right)^2 = (x+ae)^2+y^2 \tag{19}$$
$$r^2 = (x-ae)^2+y^2 \tag{20}$$

so

$$\sqrt{(x+ae)^2+y^2}+\sqrt{(x-ae)^2+y^2} = 2a \tag{21}$$

Squaring both sides, we get

$$(x+ae)^2+(x-ae)^2+2y^2+2\sqrt{(x+ae)^2+y^2}\sqrt{(x-ae)^2+y^2}=4a^2 \tag{22}$$

$$x^2+y^2+a^2e^2+\sqrt{(x+ae)^2+y^2}\sqrt{(x-ae)^2+y^2}=2a^2 \tag{23}$$

$$\sqrt{(x^2-a^2e^2)^2+y^4+y^2\left((x+ae)^2+(x-ae)^2\right)}=a^2+b^2-x^2-y^2 \tag{24}$$

$$\sqrt{y^4+2x^2y^2+x^4+2a^2e^2\left(y^2-x^2\right)+a^4e^4}=a^2+b^2-x^2-y^2 \tag{25}$$

$$y^4+2x^2y^2+x^4+2a^2e^2\left(y^2-x^2\right)+a^4e^4=\left(a^2+b^2-x^2-y^2\right)^2 \tag{26}$$

$$2x^2y^2+2\left(a^2-b^2\right)\left(y^2-x^2\right)+\left(a^2-b^2\right)^2=\left(a^2+b^2\right)^2- 2\left(a^2+b^2\right)\left(x^2+y^2\right)+2x^2y^2 \tag{27}$$

$$4b^2x^2+4a^2y^2=4a^2b^2 \tag{28}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{29}$$

To find the area of an ellipse, we can again use either polar or rectangular coordinates. To find the area in polar coordinates, consider the area of a thin wedge of radius r and angular extent $d\theta$. If r were constant over the range $\theta\in[0,2\pi]$, we'd have the area of a circle, or πr^2. The wedge has a fraction $d\theta/2\pi$ of this area, so the area of the wedge is $\frac{1}{2}r^2d\theta$, and the area of the ellipse is

$$A=\int_0^{2\pi}\frac{1}{2}r^2d\theta \tag{30}$$

Using 10 we get

$$A=\frac{b^4}{2a^2}\int_0^{2\pi}\frac{d\theta}{\left(1+e\cos\theta\right)^2} \tag{31}$$

I'm not sure how you would do this integral by hand, but for the indefinite integral Maple gives a complicated result involving arctans and tangents.

Putting in the limits, however, and restricting e to $0 \le e < 1$ gives a simple answer (using 3):

$$A = \frac{\pi b^4}{a^2 \left(1-e^2\right)^{3/2}} = \pi \frac{b^2}{\sqrt{1-e^2}} = \pi a b \tag{32}$$

Using rectangular coordinates is somewhat easier. We can work out the area for the first quadrant and multiply by 4:

$$A = 4b \int_0^a \sqrt{1-\frac{x^2}{a^2}} dx \tag{33}$$

$$= 2ab \left[\arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + x\sqrt{a^2-x^2} \right]_0^a \tag{34}$$

$$= \pi a b \tag{35}$$

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