INTENSITY AND RADIATION PRESSURE

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Intensity of radiation. Consider an elemental area dA, and suppose some radiation passes through this area at an angle θ to the normal of the area. We place a detector so that it intercepts this radiation. The energy measured by the detector within a given wavelength range $\lambda \to \lambda + d\lambda$ is given by $E_\lambda d\lambda$, and depends on the wavelength of the radiation and the time dt over which the measurement is made. It also depends on the effective area seen by the detector. Since we're viewing along a line at an angle θ relative to the normal to dA, the effective area is the projection of dA onto the plane normal to the viewing direction, which is $dA\cos\theta$. If $\theta = 0$, we're looking straight down onto dA, so the effective area is just dA itself, while if $\theta = \frac{\pi}{2}$, we're viewing along a line parallel to the plane of dA, and the effective area is just a line segment, so we won't detect anything.

The energy detected also depends on the solid angle $d\Omega$ subtended by the detector. Finally, the energy depends on what we call the *intensity* I_λ of the radiation at a given wavelength. The intensity is defined by the equation

$$E_\lambda d\lambda = I_\lambda d\lambda\ dt\ dA\cos\theta\ d\Omega \tag{1}$$

Since $E_\lambda$has units of energy per unit wavelength and λ has units of length (it's a wavelength), the quantity $E_\lambda d\lambda$ has units of energy, and the units of I_λ are, in SI units

$$[I_\lambda] = \text{J m}^{-1}\text{s}^{-1}\text{m}^{-2}\text{sr}^{-1} \tag{2}$$

The first m^{-1} comes from the wavelength interval $d\lambda$ and the m^{-2} comes from the area dA. sr stands for steradians, the units of solid angle. Sometimes we see the units written as

$$[I_\lambda] = \text{J s}^{-1}\text{m}^{-3}\text{sr}^{-1} \tag{3}$$

This can be confusing since it seems to indicate a quantity per unit volume m^{-3}, which would be some form of density. It's important to keep the m^{-1} (due to wavelength) separate from the m^{-2} (due to area).

The blackbody radiation distribution function is

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} \tag{4}$$

If we plug in the units we find that

$$[B_\lambda(T)] = \text{J sm}^2\text{s}^{-2}\text{m}^{-5} = \text{J s}^{-1}\text{m}^{-3} \tag{5}$$

Actually, the function 4 is defined to be per unit solid angle, so there is an additional unit of sr^{-1}. Steradians are like ordinary radians for measuring planar angles in the sense that they are considered 'dimensionless'. Thus $B_\lambda(T)$ has the units of intensity.

Radiation pressure. In our discussion of electromagnetic waves, we've seen that radiation exerts a pressure on any surface that it interacts with. This analysis relied on a lot of electromagnetic theory, but in fact we can use a simpler derivation to get a formula for the pressure.

Suppose a beam of radiation is reflected by a surface. The reflection is specular, meaning that the angle of incidence θ equals the angle of reflection. According to special relativity, a photon has momentum

$$p = \frac{E}{c} \tag{6}$$

When a photon is reflected from a surface, the component of momentum normal to the surface is reversed. This component is $\frac{E\cos\theta}{c}$. Using the notation above, the energy of a beam of radiation in the wavelength range $\lambda \to \lambda + d\lambda$ will therefore cause a momentum of $dp_\lambda d\lambda$ to be imparted to the surface, where we have

$$dp_\lambda d\lambda = \frac{E_\lambda d\lambda \cos\theta}{c} - \left(-\frac{E_\lambda d\lambda \cos\theta}{c}\right) \tag{7}$$

$$= 2\frac{E_\lambda d\lambda \cos\theta}{c} \tag{8}$$

In terms of intensity 1, we can write this as

$$dp_\lambda d\lambda = 2I_\lambda d\lambda\; dt\; dA \cos\theta\; d\Omega \frac{\cos\theta}{c} \tag{9}$$

$$= 2I_\lambda d\lambda\; dt\; dA\; d\Omega \frac{\cos^2\theta}{c} \tag{10}$$

Note that we have two separate factors of $\cos\theta$ here: one from the effective area seen by a detector and one from the fraction of the radiation's momentum that is reflected by the surface.

To convert this to a pressure, we use the fact that pressure is force per unit area, and that force is the rate of change of momentum. We also need to integrate over all solid angles, since in general, photons can arrive from any direction. Therefore

$$P_\lambda d\lambda = \left[\int \frac{dp_\lambda}{dt\,dA} d\Omega\right] d\lambda \tag{11}$$

$$= \left[\int 2I_\lambda \frac{\cos^2\theta}{c} d\Omega\right] d\lambda \tag{12}$$

In general, the intensity function I_λ can vary over direction, but for a blackbody, it is isotropic so we can take it outside the integral. In spherical coordinates

$$d\Omega = \sin\theta\; d\theta\; d\phi \tag{13}$$

so we have, for blackbody radiation

$$P_\lambda d\lambda = \frac{2I_\lambda}{c} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\; \cos^2\theta \sin\theta \tag{14}$$

$$= \frac{8\pi I_\lambda}{3c} \tag{15}$$

For a blackbody, we can integrate over wavelength by setting $I_\lambda = B_\lambda(T)$ from 4. Using Maple to do the integral, we have

$$P = \frac{8\pi}{3c} \int_0^\infty \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} d\lambda \tag{16}$$

$$= \frac{16\pi^5 (kT)^4}{45 (hc)^3} \tag{17}$$

The Stefan-Boltzmann constant can be expressed in terms of more fundamental constants as

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3} \tag{18}$$

so we have

$$P = \frac{8}{3c}\sigma T^4 \tag{19}$$

Comparing with formula for the energy density of a blackbody, we see that

$$u = \frac{4}{c}\sigma T^4 \tag{20}$$

so that the pressure for a fully reflecting blackbody is

$$P=\frac{2}{3}u \tag{21}$$

A blackbody is usually defined as an object that absorbs all wavelengths of light, in which case the pressure would be half that above since the momentum is merely absorbed rather than reversed. In that case we have

$$P=\frac{u}{3} \tag{22}$$

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