

ORBITAL VELOCITIES AT PERIHELION AND APHELION

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The angular momentum of a two-body system is

$$\mathbf{L} = \mu \mathbf{r} \times \mathbf{v} \quad (1)$$

where μ is the reduced mass. At aphelion and perihelion, $\mathbf{r} \perp \mathbf{v}$ and since \mathbf{L} is conserved at all points in the orbit

$$\mu r_a v_a = \mu r_p v_p \quad (2)$$

From the polar equation for an ellipse, with r the distance from a focus, and $\theta = 0$ for perihelion and $\theta = \pi$ for aphelion:

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad (3)$$

$$r_a = \frac{a(1 - e^2)}{1 - e} = a(1 + e) \quad (4)$$

$$r_p = \frac{a(1 - e^2)}{1 + e} = a(1 - e) \quad (5)$$

so the ratio of the velocities is

$$\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1 + e}{1 - e} \quad (6)$$

The energy of the system is

$$E = \frac{\mu}{2} v^2 - \frac{GM\mu}{r} \quad (7)$$

Since E is conserved, we can equate the energies at aphelion and perihelion to get

$$\frac{v_a^2}{2} - \frac{GM}{r_a} = \frac{v_p^2}{2} - \frac{GM}{r_p} \quad (8)$$

From 6 we get

$$v_p = \frac{1+e}{1-e}v_a \quad (9)$$

$$\left[\left(\frac{1+e}{1-e} \right)^2 - 1 \right] v_a^2 = 2GM \left(\frac{1}{r_p} - \frac{1}{r_a} \right) \quad (10)$$

$$\frac{4e}{(1-e)^2}v_a^2 = 2GM \frac{r_a - r_p}{r_a r_p} \quad (11)$$

$$= \frac{4GMae}{a^2(1-e^2)} \quad (12)$$

$$v_a^2 = \frac{GM}{a} \frac{1-e}{1+e} \quad (13)$$

where we used 4 and 5 to get the fourth line. Substituting 9 into the last line we get

$$v_p^2 = \frac{GM}{a} \frac{1+e}{1-e} \quad (14)$$

From 1 at aphelion, 4 and 13 we get

$$L = \mu r_a v_a = \mu a (1+e) \sqrt{\frac{GM}{a} \frac{1-e}{1+e}} = \mu \sqrt{GMa(1-e^2)} \quad (15)$$