ORBITS IN THE CENTRE OF MASS FRAME - ENERGY AND ANGULAR MOMENTUM

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For two masses m_1 and m_2 interacting via gravity, it's easiest to do calculations in the centre of mass frame. The position of the centre of mass is defined as

$$\mathbf{R}\equiv\frac{m_1\mathbf{r}_1'+m_2\mathbf{r}_2'}{m_1+m_2} \tag{1}$$

where $\mathbf{r}_i'$ is the position of m_i in a coordinate system where the origin could be anywhere.

In the centre of mass frame, we take the centre of mass to be at the origin, so that $\mathbf{R}=0$, giving

$$\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}=0 \tag{2}$$

where now $\mathbf{r}_i$ (without the prime) indicates the position relative to the centre of mass. If we introduce the relative position

$$\mathbf{r}\equiv\mathbf{r}_2-\mathbf{r}_1 \tag{3}$$

then

$$\frac{m_1\mathbf{r}_1+m_2\left(\mathbf{r}+\mathbf{r}_1\right)}{m_1+m_2}=0 \tag{4}$$

$$\mathbf{r}_1=-\frac{m_2}{m_1+m_2}\mathbf{r} \tag{5}$$

$$\frac{m_1\left(\mathbf{r}_2-\mathbf{r}\right)+m_2\mathbf{r}_2}{m_1+m_2}=0 \tag{6}$$

$$\mathbf{r}_2=\frac{m_1}{m_1+m_2}\mathbf{r} \tag{7}$$

In terms of the *reduced mass*

$$\mu\equiv\frac{m_1m_2}{m_1+m_2} \tag{8}$$

we get

$$\mathbf{r}_1 = -\frac{\mu}{m_1}\mathbf{r} \tag{9}$$

$$\mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r} \tag{10}$$

The velocities of the two masses are the derivatives of the positions, so that

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} \tag{11}$$

and the rate of change of the relative position is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{12}$$

The total energy of the two mass system is the sum of the two kinetic energy terms and the gravitational potential energy term, so

$$E = \frac{1}{2}m_1\left|\mathbf{v}_1\right|^2 + \frac{1}{2}m_2\left|\mathbf{v}_2\right|^2 - G\frac{m_1 m_2}{r} \tag{13}$$

where

$$r = \left|\mathbf{r}_2 - \mathbf{r}_1\right| \tag{14}$$

The gravitational potential energy can be written in terms of the reduced mass and the total mass $M \equiv m_1 + m_2$ as

$$-G\frac{m_1 m_2}{r} = -G\left(m_1 + m_2\right)\frac{\mu}{r} = -G\frac{M\mu}{r} \tag{15}$$

The kinetic energy can be rewritten by using 9 and 10

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} \tag{16}$$

$$= -\frac{\mu}{m_1}\mathbf{v} \tag{17}$$

$$\mathbf{v}_2 = \frac{\mu}{m_2}\mathbf{v} \tag{18}$$

$$\frac{1}{2}m_1\left|\mathbf{v}_1\right|^2 + \frac{1}{2}m_2\left|\mathbf{v}_2\right|^2 = \frac{1}{2}\frac{\mu^2}{m_1}v^2 + \frac{1}{2}\frac{\mu^2}{m_2}v^2 \tag{19}$$

$$= \frac{1}{2}\mu^2\frac{m_1 + m_2}{m_1 m_2}v^2 \tag{20}$$

$$= \frac{1}{2}\mu v^2 \tag{21}$$

Thus the total energy is

$$E = \frac{1}{2}\mu v^2 - G\frac{M\mu}{r} \tag{22}$$

which is the energy of a mass μ moving around another mass M, the latter of which is fixed at the origin.

The orbital angular momentum (that is, the angular momentum due to the masses orbiting about each other, not including any angular momentum due to each mass rotating on its own axis) is

$$\mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2 \tag{23}$$

$$= -m_1\frac{\mu}{m_1}\mathbf{r} \times \left(-\frac{\mu}{m_1}\mathbf{v}\right) + m_2\frac{\mu}{m_2}\mathbf{r} \times \left(\frac{\mu}{m_2}\mathbf{v}\right) \tag{24}$$

$$= \mu^2\mathbf{r} \times \mathbf{v}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \tag{25}$$

$$= \frac{m_1 + m_2}{m_1 m_2}\mu^2\mathbf{r} \times \mathbf{v} \tag{26}$$

$$= \mu\mathbf{r} \times \mathbf{v} \tag{27}$$

Thus the angular momentum is due to the reduced mass alone, which is consistent with this mass orbiting about a fixed mass at the origin.

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