UNCERTAINTY PRINCIPLE - A COUPLE OF EXAMPLES FROM ASTRONOMY

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The uncertainty principle relates the standard deviations of two observables A and B to the expectation value of their commutator:

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2 \tag{1}$$

For position x and momentum p, $[x,p]=i\hbar$ so

$$\sigma_x\sigma_p \geq \frac{\hbar}{2} \tag{2}$$

Example 1. In white dwarf stars, atoms become crushed together so that electrons and protons are much closer to each other than in ordinary hydrogen gas, where the mean radius of the electron's orbit is the Bohr radius of $a = 5.29\times10^{-11}$ m. In a white dwarf, this distance gets compressed to around $\sigma_x \approx 1.5\times10^{-12}$ m. With the electron's location thus localized, its momentum must be uncertain by an amount

$$\sigma_p \approx \frac{\hbar}{2\times1.5\times10^{-12}} = 3.5\times10^{-23}\text{ kg m s}^{-1} \tag{3}$$

The minimum average momentum must be equal to σ_p (otherwise the range of values for p would include negative values) so the electron's minimum speed (assuming non-relativistic speeds) is

$$v_{min} = \frac{\sigma_p}{m_e} = \frac{3.5\times10^{-23}}{9.10938291\times10^{-31}} = 3.86\times10^{7}\text{ m s}^{-1} \tag{4}$$

This is about $0.13c$ so we should probably use relativity to calculate a better value, but at least it gives an idea of how fast electrons must be moving in a white dwarf.

The energy-time uncertainty relation is a bit more subtle, since time in non-relativistic quantum mechanics is not an observable property of a quantum state; rather it's a background parameter on which the quantum state depends. The energy-time uncertainty relation is usually given as

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{5}$$

where

$$\Delta t \equiv \frac{\sigma_Q}{|d\langle Q\rangle /dt|} \tag{6}$$

with Q representing some arbitrary observable on the system. That is, to first order, Δt is the time interval during which $\langle Q\rangle$ changes by one standard deviation. For a time-independent hamiltonian and a given set of initial conditions, the probabilities of finding the system in any given energy state do not depend on time, so ΔE is constant, and serves as a constraint on the time scale over which other observables Q can change.

Conversely, if we can measure Δt for some observable (that is, if we can measure how fast some parameter of the system changes), we can get an estimate of ΔE.

Example 2. An electron in the first excited state decays to the ground state in a time interval of around 10^{-8} s, by emitting a photon. Since such a decay is a change in an observable property of the system, we can use this time as an estimate of Δt and use it to derive an estimate of ΔE, the standard deviation of the excited state energy.

$$\Delta E \approx \frac{\hbar}{2\Delta t} = 5.27 \times 10^{-27}\text{ J} = 3.29 \times 10^{-8}\text{ eV} \tag{7}$$

This gives rise to a spread of wavelengths for the emitted photon:

$$E = h\nu = \frac{hc}{\lambda} \tag{8}$$

$$|\Delta E| = hc\frac{\Delta\lambda}{\lambda^2} \tag{9}$$

The transition $2 \to 1$ is the first spectral line in the Lyman series with a wavelength of $\lambda_{2\to 1} = 121.6$ nm so, using $hc = 1240$ eV nm we have

$$\Delta\lambda = 3.29 \times 10^{-8}\frac{(121.6)^2}{1240} = 3.92 \times 10^{-7}\text{ nm} \tag{10}$$

This *natural broadening* of spectral lines would seem to be negligible.

Incidentally, this shows the inadequacy of the Schrödinger equation for studying the dynamics of electron energy levels, since the excited states of an atom are all eigenstates of the hamiltonian and thus should be stable. The fact that spontaneous decay occurs illustrates the need for quantum electrodynamics.