VELOCITY IN AN ELLIPTICAL ORBIT

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We did a sample calculation of the Sun-Jupiter system using Kepler's laws by approximating the orbits of the Sun and Jupiter about the centre of mass by circles. We can, however, derive equations for the radial and tangential velocity components for the correct case of elliptical orbits. We can start with the polar equation of an ellipse:

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{1}$$

The velocity of an object in polar coordinates is

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} \tag{2}$$

$$= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{3}$$

Differentiating 1 with respect to time we get

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \tag{4}$$

$$= \frac{ae\left(1-e^2\right)\sin\theta}{\left(1+e\cos\theta\right)^2}\dot{\theta} \tag{5}$$

In deriving Kepler's laws, we got an expression for the total angular momentum of the system:

$$L = \mu\sqrt{GMa\left(1-e^2\right)} \tag{6}$$

In vector form, the angular momentum can be expressed in terms of the velocity of the centre of mass as

$$\mathbf{L} = \mu\mathbf{r}\times\mathbf{v} \tag{7}$$

$$= \mu\mathbf{r}\times\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) \tag{8}$$

$$= \mu r^2\dot{\theta}\hat{\mathbf{z}} \tag{9}$$

since $\boldsymbol{r}\times\hat{\boldsymbol{r}}=0$ as these two vectors are parallel, and $\boldsymbol{r}\times\hat{\boldsymbol{\theta}}=r\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\theta}}=r\hat{\boldsymbol{z}}$. Here $\hat{\boldsymbol{z}}$ is a unit vector perpendicular to the plane of the orbit.

Therefore

$$\dot{\theta}=\frac{\sqrt{GMa\left(1-e^{2}\right)}}{r^{2}} \tag{10}$$

Substituting 1 into this, we get

$$\dot{\theta}=\frac{\sqrt{GMa\left(1-e^{2}\right)}\left(1+e\cos\theta\right)^{2}}{a^{2}\left(1-e^{2}\right)^{2}} \tag{11}$$

From Kepler's third law relating the period P of the orbit to the semimajor axis a:

$$P^{2}=\frac{4\pi^{2}}{GM}a^{3} \tag{12}$$

we can eliminate GM to get

$$\dot{\theta}=\frac{\sqrt{4\pi^{2}a^{4}\left(1-e^{2}\right)}\left(1+e\cos\theta\right)^{2}}{a^{2}\left(1-e^{2}\right)^{2}P} \tag{13}$$

$$=\frac{2\pi\left(1+e\cos\theta\right)^{2}}{P\left(1-e^{2}\right)^{3/2}} \tag{14}$$

From this we can get the velocity components:

$$v_{\theta}=r\dot{\theta} \tag{15}$$

$$=\frac{a\left(1-e^{2}\right)}{1+e\cos\theta}\frac{2\pi\left(1+e\cos\theta\right)^{2}}{P\left(1-e^{2}\right)^{3/2}} \tag{16}$$

$$=\frac{2\pi a\left(1+e\cos\theta\right)}{P\sqrt{1-e^{2}}} \tag{17}$$

$$v_{r}=\frac{dr}{d\theta}\dot{\theta} \tag{18}$$

$$=\frac{ae\left(1-e^{2}\right)\sin\theta}{\left(1+e\cos\theta\right)^{2}}\frac{2\pi\left(1+e\cos\theta\right)^{2}}{P\left(1-e^{2}\right)^{3/2}} \tag{19}$$

$$=\frac{2\pi ae\sin\theta}{P\sqrt{1-e^{2}}} \tag{20}$$

The square magnitude of the velocity is then

$$\begin{aligned}
v^2 &= v_\theta^2 + v_r^2 && (21)\\
&= \frac{4\pi^2 a^2}{p^2\left(1-e^2\right)}\left(1+2e\cos\theta+e^2\left(\sin^2\theta+\cos^2\theta\right)\right) && (22)\\
&= \frac{4\pi^2 a^2\left(1+e^2+2e\cos\theta\right)}{P^2\left(1-e^2\right)} && (23)
\end{aligned}$$

From 12 we get

$$v^2 = \frac{GM\left(1+e^2+2e\cos\theta\right)}{a\left(1-e^2\right)} \tag{24}$$

From 1 we have

$$\begin{aligned}
\frac{2}{r}-\frac{1}{a} &= \frac{2\left(1+e\cos\theta\right)}{a\left(1-e^2\right)}-\frac{1}{a} && (25)\\
&= \frac{2\left(1+e\cos\theta\right)-\left(1-e^2\right)}{a\left(1-e^2\right)} && (26)\\
&= \frac{1+e^2+2e\cos\theta}{a\left(1-e^2\right)} && (27)
\end{aligned}$$

Therefore, comparing with 24 we get

$$v^2 = GM\left(\frac{2}{r}-\frac{1}{a}\right) \tag{28}$$

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