

VIRIAL THEOREM FOR GRAVITATIONAL ORBITS

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We've already met the virial theorem in quantum mechanics where it had the form, in one dimension:

$$2\langle K \rangle = \left\langle x \frac{dU}{dx} \right\rangle \quad (1)$$

where K is the kinetic energy and U is the potential energy. Applying this to an object of mass m in a gravitational orbit about another mass M ,

$$U = -\frac{GMm}{r} \quad (2)$$

so using r as our one position dimension, we get

$$2\langle K \rangle = \left\langle r \frac{dU}{dr} \right\rangle = GMm \left\langle \frac{1}{r} \right\rangle = -\langle U \rangle \quad (3)$$

For those interested in a completely classical derivation of this result, full details are given in Carroll & Ostlie, chapter 2.

To use this formula, we need to find $\langle U \rangle$, which requires us to average the potential energy over time. Since the motion is periodic, we can find the average over a single orbit, that is, over the time interval $t \in [0, P]$. We have

$$\langle U \rangle = -\frac{GMm}{P} \int_0^P \frac{dt}{r} \quad (4)$$

For an elliptical orbit (remember that, in polar coordinates, r is the distance of the planet from the *focus* of the ellipse, not the centre):

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad (5)$$

so we can transform the integral to

$$\langle U \rangle = -\frac{GMm}{Pa(1-e^2)} \int_0^P (1+e\cos\theta) dt \quad (6)$$

$$= -\frac{GMm}{Pa(1-e^2)} \int_0^{2\pi} (1+e\cos\theta) \frac{dt}{d\theta} d\theta \quad (7)$$

$$= -\frac{GMm}{Pa(1-e^2)} \int_0^{2\pi} \frac{1+e\cos\theta}{\dot{\theta}} d\theta \quad (8)$$

We've already worked out $\dot{\theta}$ as

$$\dot{\theta} = \frac{2\pi(1+e\cos\theta)^2}{P(1-e^2)^{3/2}} \quad (9)$$

so we have to evaluate the integral

$$\int_0^{2\pi} \frac{1+e\cos\theta}{\dot{\theta}} d\theta = \int_0^{2\pi} \frac{1+e\cos\theta}{2\pi(1+e\cos\theta)^2} P(1-e^2)^{3/2} d\theta \quad (10)$$

$$= \frac{P(1-e^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{1+e\cos\theta} \quad (11)$$

Carroll & Ostlie give the answer for the integral, but for those interested, the indefinite integral is (using Maple):

$$\int \frac{d\theta}{1+e\cos\theta} = \frac{2}{\sqrt{1-e^2}} \arctan\left(\frac{(1-e)\tan(\theta/2)}{\sqrt{1-e^2}}\right) \quad (12)$$

Evaluating the limits we get

$$\int_0^{2\pi} \frac{d\theta}{1+e\cos\theta} = \frac{2\pi}{\sqrt{1-e^2}} \quad (13)$$

so

$$\int_0^{2\pi} \frac{1+e\cos\theta}{\dot{\theta}} d\theta = P(1-e^2) \quad (14)$$

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{a} \quad (15)$$

$$\langle U \rangle = -\frac{GMm}{a} \quad (16)$$

Curiously, although $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a}$, it turns out that $\langle r \rangle \neq a$. Using the same technique as above, we have

$$\langle r \rangle = \frac{a(1-e^2)}{P} \int_0^P \frac{dt}{1+e\cos\theta} \quad (17)$$

$$= \frac{a(1-e^2)^{5/2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^3} \quad (18)$$

The integral can be evaluated in Maple (by imposing the constraint $0 < e < 1$). For reference, the indefinite integral is (using Maple):

$$\int \frac{d\theta}{(1+e\cos\theta)^3} = \left[\frac{\sqrt{-e^2+1} e \sin(\theta) (e^2 - 3e \cos(\theta) - 4)}{2} - \frac{(e^2 + 2) (1 + e \cos(\theta))^2 \arctan\left(\frac{(e-1)(\csc(\theta) - \cot(\theta))}{\sqrt{-e^2+1}}\right)}{\sqrt{-e^2+1}} \right] \div \left[\sqrt{-e^2+1} (e-1)^2 (e+1)^2 (1+e\cos(\theta))^2 \right] \quad (19)$$

Inserting the limits simplifies things considerably:

$$\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^3} = \frac{\pi(2+e^2)}{(1-e^2)^{5/2}} \quad (20)$$

so we get

$$\langle r \rangle = \frac{a}{2} (2+e^2) \quad (21)$$

For a circle $e = 0$ and $\langle r \rangle = a$, but for any ellipse with a non-zero eccentricity, $\langle r \rangle > a$. Given Kepler's second law (equal areas in equal times), this makes sense in that, since the planet moves more slowly when r is larger, it spends more time at greater distances. However, this doesn't explain why $\langle \frac{1}{r} \rangle = \frac{1}{a}$; using this logic you'd expect $\langle \frac{1}{r} \rangle < \frac{1}{a}$.

PINGBACKS

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