ANGULAR DISTANCES ON THE CELESTIAL SPHERE

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 1, Problems 1.8 - 1.11.

Sometimes we need to know the angular distance between two points on the celestial sphere. By *angular distance*, I mean the angle subtended by lines drawn from the centre of the Earth to each of the two points. For example, the angular distance between the north pole and any point on the equator is 90°. The angular diameters of both the moon and the sun are around 30′ or half a degree. One practical application of such a calculation is to determine if two stars that are close to each other in the sky will both be visible within the field of view of a telescope using a particular eyepiece, since an eyepiece's field of view is typically given as an angle.

Since the coordinates of objects are typically given in right ascension (RA) and declination (Dec), we'd like a formula that gives the angular distance between two sets of such coordinates. The derivation of the formula uses spherical trigonometry and is given in full in Carroll & Ostlie's Section 1.3 (in particular Fig. 1.17 and surrounding text), so we won't go through the details here. The exact formulas relating the angular distance $\Delta\theta$ between a point P with coordinates $(RA, Dec) = (\alpha, \delta)$ and an adjacent point Q with coordinates $(\alpha + \Delta\alpha, \delta + \Delta\delta)$ are

$$\sin(\Delta\alpha)\cos(\delta + \Delta\delta) = \sin(\Delta\theta)\sin\phi \tag{1}$$

$$\cos[90^\circ - (\delta + \Delta\delta)] = \cos(90^\circ - \delta)\cos(\Delta\theta) + \sin(90^\circ - \delta)\sin(\Delta\theta)\cos\phi \tag{2}$$

where ϕ is the angle, measured counterclockwise as we look at the sky, between a line due north from P and the line PQ (by 'line' I mean a segment of a great circle, since these lines are drawn on a sphere), measured . In principle, ϕ is of course determined by P and Q so we should be able to eliminate it from these equations, but it turns out that P and Q are quite close together, there is an easier way to eliminate ϕ, as we'll see now.

If we assume that $\Delta\delta$ and $\Delta\alpha$ are both small enough, we can use the approximations $\sin x \approx x$ and $\cos x \approx 1$ together with $\cos(90^\circ - x) = \sin x$, $\sin(90^\circ - x) = \cos x$ and the trig formulas for the sum and difference of angles to simplify the above formulas. From 1 we get, by keeping only first order terms in $\Delta\delta$ and $\Delta\alpha$:

$$\begin{aligned} \Delta\alpha\left(\cos\delta - \sin\Delta\delta\sin\delta\right) &= \Delta\theta\sin\phi && (3) \\ \Delta\alpha\cos\delta &= \Delta\theta\sin\phi && (4) \end{aligned}$$

From 2 we get

$$\begin{aligned} \sin\left(\delta + \Delta\delta\right) &= \sin\delta + \Delta\theta\cos\delta\cos\phi && (5) \\ \sin\delta + \Delta\delta\cos\delta &= \sin\delta + \Delta\theta\cos\delta\cos\phi && (6) \\ \Delta\delta\cos\delta &= \Delta\theta\cos\delta\cos\phi && (7) \\ \Delta\delta &= \Delta\theta\cos\phi && (8) \end{aligned}$$

We can now square 4 and 8 and add them to get

$$\boxed{(\Delta\theta)^2 = (\Delta\alpha\cos\delta)^2 + (\Delta\delta)^2} \quad (9)$$

Note that the approximation $\sin x \approx x$ assumes that x is in radians, not degrees. However, because each term in this equation contains the square of an angle, the conversion factor from degrees to radians cancels out, so the formula is valid whether we specify angles in degrees or radians (as long as you remember to take δ in the correct units when taking the cosine).

Example 1. The closest star system to the Sun is the α Centauri triple star system. Of the three stars in this system, Proxima Centauri (α Centauri C) is the closest to us. The coordinates of Proxima Centauri in the year 2000 (known as the epoch J2000.0) were $(\alpha,\delta)_C = \left(14^h 29^m 42.95^s, -62^\circ 40' 46.1''\right)$ (unfortunately for those of us in the far northern hemisphere, the α Centauri system's extreme southern declination means we can never see it). The brightest star in the system (α Centauri A) has J2000.0 coordinates of $(\alpha,\delta)_A = \left(14^h 39^m 36.50^s, -60^\circ 50' 02.3''\right)$. Since these two points are quite close to each other, we can use 9 to see how far apart they appear in the sky. We can first convert the coordinates into degrees in decimal form. The conversion for α is $1^h = 15^\circ$, so 1^m is 1/60 of 15° or $15' = 0.25^\circ$. Likewise, $1^s = 15'' = 15/3600^\circ$. Therefore $14^h 29^m 42.95^s = 14 \times 15 + 29 \times 0.25 + 42.95 \times 15/3600 = 217.42896^\circ$.

$$\begin{aligned} (\alpha,\delta)_C &= (217.42896^\circ, -62.67947^\circ) && (10) \\ (\alpha,\delta)_A &= (219.90208^\circ, -60.83397^\circ) && (11) \end{aligned}$$

We therefore have

$$\Delta\alpha = 2.47312^\circ \tag{12}$$
$$\Delta\delta = 1.8455^\circ \tag{13}$$
$$\cos\delta = \cos(-62.67947^\circ) \tag{14}$$
$$= 0.45897 \tag{15}$$
$$\Delta\theta = \sqrt{(\Delta\alpha\cos\delta)^2 + (\Delta\delta)^2} \tag{16}$$
$$= 2.1666^\circ \tag{17}$$

If the distance to the α Centauri system is $r = 4.0 \times 10^{16}$ m (and we assume that all members of the system are the same distance from us, or at least that the radial distances between the three stars is negligible as a fraction of their distance from us), then Proxima Centauri and α Centauri A are separated by

$$d_{C\to A} = r\Delta\theta \tag{18}$$

This time, we do have to use radians, so we get

$$d_{C\to A} = \left(4.0 \times 10^{16}\right) \frac{\pi}{180} (2.1666) = 1.512 \times 10^{15} \text{ m} \tag{19}$$

This is about 3.78% of the system's distance from us.

Example 2. ***Proper motion and precession*****.** The position of a star as seen from Earth can change due to two effects: the star's actual motion through space, known as *proper motion*, and the precession of the Earth's axis. The latter effect is due to the direction of the Earth's axis slowly changing so that the north celestial pole moves through a circle in the sky with a period of about 26,000 years. Both these effects cause a star's coordinates to change. In the previous example, we gave the coordinates of the α Centauri system for epoch J2000.0. Astronomical coordinates are typically updated for the effects of precession every 50 years, but of course it is sometimes necessary to know the precise coordinates of a star for a time between updates. Calculating these positions exactly is a complicated business, but there are approximate formulas that are usually good enough. The changes in coordinates due to precession relative to J2000.0 are given by

$$\Delta\alpha = M + N\sin\alpha\tan\delta \tag{20}$$
$$\Delta\delta = N\cos\alpha \tag{21}$$

where

$$M = 1^\circ.2812323T + 0^\circ.0003879T^2 + 0^\circ.0000101T^3 \tag{22}$$
$$N = 0^\circ.5567530T - 0^\circ.0001185T^2 - 0^\circ.0000116T^3 \tag{23}$$
$$T = (t - 2000.0)/100 \tag{24}$$

and t is the current date, specified as the current year plus a fraction for the current date.

For Proxima Centauri in the year 2010, using these formulas gives

$$\Delta\alpha = 0^\circ.1936 \tag{25}$$
$$\Delta\delta = -0^\circ.0442 \tag{26}$$

Substituting into 9 we get

$$\Delta\theta_{precession} = 0^\circ.0992 \tag{27}$$
$$= 357'' \approx 6' \tag{28}$$

The proper motion of Proxima Centauri is measured to be $3.84''$ per year with a position angle of $\phi = 282^\circ$. Over 10 years, proper motion results in a shift of $\Delta\theta_{pm} = 38.4''$, so the precession effect is larger. We can get the shifts in RA and Dec from 4 and 8:

$$\Delta\alpha_{pm} = \Delta\theta_{pm}\frac{\sin\phi}{\cos\delta} \tag{29}$$
$$= -81.84'' = -5.46^s \tag{30}$$
$$\Delta\delta_{pm} = \Delta\theta_{pm}\cos\phi \tag{31}$$
$$= 7.98'' \tag{32}$$