KEPLER'S LAWS

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problem 2.6.

The total angular momentum of a two-body system interacting under gravity is

$$\mathbf{L}=\mu\mathbf{r}\times\mathbf{v} \tag{1}$$

where μ is the reduced mass

$$\mu=\frac{m_1m_2}{m_1+m_2} \tag{2}$$

and $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ is the position of m_2 relative to m_1, and $\mathbf{v}=d\mathbf{r}/dt$. An important conservation law follows from taking the derivative with respect to time:

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \mu\frac{d\mathbf{r}}{dt}\times\mathbf{v}+\mu\mathbf{r}\times\frac{d\mathbf{v}}{dt} && (3)\\ &= \mu\mathbf{v}\times\mathbf{v}+\mathbf{r}\times\mathbf{F} && (4)\\ &= 0 && (5)\end{aligned}$$

where $\mathbf{r}\times\mathbf{F}=0$ because gravity acts along the direction separating the two masses, so $\mathbf{r}\parallel\mathbf{F}$. Thus for any two objects in orbit about their centre of mass, angular momentum is conserved.

Using this fact and Newton's gravitational force law, Carroll & Ostlie derive (in section 2.3) a formula for r as a function of the angle θ measured from the perihelion point on the major axis:

$$\boxed{r=\frac{L^2/\mu^2}{GM\left(1+e\cos\theta\right)}} \tag{6}$$

where $M=m_1+m_2$.

Comparing this with the polar equation for an ellipse:

$$r=\frac{a\left(1-e^{2}\right)}{1+e\cos\theta} \tag{7}$$

we see that if the angular momentum is

$$L=\mu\sqrt{GMa\left(1-e^{2}\right)} \tag{8}$$

then 6 is the equation of an ellipse. This is *Kepler's first law*, derived from Newtonian mechanics.

To translate this to motions of the two masses separately, we can use the relations between the position vectors of the masses and **r**.

$$\mathbf{r}_{1} = -\frac{\mu}{m_{1}}\mathbf{r} \tag{9}$$

$$\mathbf{r}_{2} = \frac{\mu}{m_{2}}\mathbf{r} \tag{10}$$

Substituting these into 6 we get

$$r_{1} = \frac{L^{2}}{GMm_{1}\mu\left(1+e\cos\theta\right)} \tag{11}$$

$$r_{2} = \frac{L^{2}}{GMm_{2}\left(1+e\cos\theta\right)} \tag{12}$$

These equations still have the form of ellipses, so we see that both masses are in elliptical orbits about the centre of mass, which itself is fixed at one focus of the ellipse. Since the eccentricity e is the same for both masses and L is constant (it's the *total* angular momentum of the two masses together), we can get the semimajor axis for each mass from

$$\frac{L^{2}}{GMm_{i}\mu} = a_{i}\left(1-e^{2}\right) \tag{13}$$

$$a_{i} = \frac{L^{2}}{GMm_{i}\mu\left(1-e^{2}\right)} \tag{14}$$

That is, the semimajor axis is inversely proportional to the mass of the object.

The constancy of angular momentum can also be used to derive *Kepler's second law*, which states that the rate at which the radius vector of each mass sweeps out area is a constant. Carroll & Ostlie give the details, with the result

$$\boxed{\frac{dA}{dt} = \frac{L}{2\mu}} \tag{15}$$

where A is the area of the sector of the ellipse between the perihelion and the vector $\mathbf{r}$ at time t. This result pertains to the reduced mass, but given that

$$\frac{L}{\mu} = |\mathbf{r} \times \mathbf{v}| \tag{16}$$

we can follow through the same derivation in Carroll & Ostlie to get the equation for each mass:

$$\frac{dA_i}{dt} = \frac{L_i}{2m_i} \tag{17}$$

where

$$\mathbf{L}_i = m_i \mathbf{r}_i \times \mathbf{v}_i \tag{18}$$

By substituting 9 and 10 into 3 we can see that the angular momentum of each mass separately is also conserved, so the equal areas in equal times law applies to each mass separately as well.

At perihelion and aphelion, $\mathbf{r}_i$ and $\mathbf{v}_i$ are perpendicular so at these points (for both masses):

$$L_i = m_i r_{i(a,p)} v_{i(a,p)} \tag{19}$$

and for the centre of mass

$$L = \mu r_{a,p} v_{a,p} \tag{20}$$

For any ellipse

$$r_p = a(1-e) \tag{21}$$
$$r_a = a(1+e) \tag{22}$$

so for the centre of mass, using 8

$$\begin{aligned} v_a &= \frac{L}{\mu r_a} && (23) \\ &= \frac{\mu\sqrt{GMa(1-e^2)}}{\mu a(1+e)} && (24) \\ &= \sqrt{\frac{GM(1-e)}{a(1+e)}} && (25) \\ v_p &= \sqrt{\frac{GM(1+e)}{a(1-e)}} && (26) \end{aligned}$$

Finally, we can get Kepler's third law by integrating 15 over one orbital period P to get

$$A = \frac{L}{2\mu}P \tag{27}$$

Since the area of an ellipse is $A = \pi ab = \pi a^2\sqrt{1-e^2}$ we have

$$\begin{aligned} \pi a^2\sqrt{1-e^2} &= \frac{1}{2}\sqrt{GMa(1-e^2)}P && (28) \\ P^2 &= \frac{4\pi^2}{GM}a^3 && (29) \end{aligned}$$

The square of the period is proportional to the cube of the semimajor axis.

Example. If we look at the Sun and Jupiter (ignoring all the other planets), we can plug in a few numbers. The eccentricity of Jupiter's orbit is $e = 0.048$, its period is $P = 11.86$ yr, its mass is $0.0009546M_S$ and its semimajor axis is $a = 5.2$ AU. In these units, $4\pi^2/GM = 1$, so from 8, the total angular momentum of the system is

$$L = 2\pi\mu\sqrt{a(1-e^2)} \tag{30}$$

The reduced mass is

$$\mu = \frac{0.0009546M_S^2}{1.0009546M_S} = 0.000954M_S \tag{31}$$

so

$$L = 0.01367M_S\ \text{AU}^2\text{yr}^{-1} \tag{32}$$

To find the contribution of the Sun to L, we need its semimajor axis, which we can get from 14

$$\begin{aligned} a_S &= \frac{(0.01365M_S)^2}{4\pi^2\,(0.000945)\,M_S^2\,(1-e^2)} && (33) \\ &= 0.005 \text{ AU} && (34) \end{aligned}$$

In order to calculate L_S from 19, we need the velocity v at aphelion or perihelion, which at present we don't know. However, if we approximate the sun's orbit by a circle, then v is constant for the orbit and we have

$$v_S = \frac{2\pi a_S}{P} = 0.00265 \text{ AU yr}^{-1} \tag{35}$$

Then the Sun's angular momentum is

$$L_S = M_S a_S v_S = 1.3 \times 10^{-5} M_S \text{ AU}^2\text{yr}^{-1} \tag{36}$$

Doing the same calculation for Jupiter, we find

$$\begin{aligned} v_J &= \frac{2\pi a}{P} = 2.755 \text{ AU yr}^{-1} && (37) \\ L_J &= 0.0009546M_S\,(5.2)\,(2.755) && (38) \\ &= 0.01367M_S \text{ AU}^2 \text{ yr}^{-1} && (39) \end{aligned}$$

Essentially all of the orbital angular momentum is due to Jupiter.

The rotational angular momentum of a solid sphere is

$$\begin{aligned} L &= I\omega && (40) \\ &= \frac{2}{5}mr^2\omega && (41) \end{aligned}$$

where I is the moment of inertia, r is the radius of the sphere and ω is the angular velocity. Although neither the Sun nor Jupiter is a solid sphere, we can get approximate values by assuming they are. The rotation period D_S of the Sun is around 26 days or 0.0712 years. Jupiter's 'day' is only about 10 hours, or 0.00114 years. The angular velocities are

$$\begin{aligned} \omega_S &= \frac{2\pi}{D_S} = 87.88 \text{ yr}^{-1} && (42) \\ \omega_J &= \frac{2\pi}{D_J} = 5511 \text{ yr}^{-1} && (43) \end{aligned}$$

and the radii are

$$r_S = 0.00465 \text{ AU} \tag{44}$$
$$r_J = 0.00047 \text{ AU} \tag{45}$$

giving

$$L_S = \frac{2}{5}M_S r_S^2 \omega_S = 7.6\times 10^{-4} M_S \text{ AU}^2\text{yr}^{-1} \tag{46}$$
$$L_J = \frac{2}{5}(0.0009546 M_S) r_J^2 \omega_J = 4.65\times 10^{-7} M_S \text{ AU}^2 \text{ yr}^{-1} \tag{47}$$

Virtually all the angular momentum in the Sun-Jupiter system is due to the orbital motion of Jupiter.

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