VIRIAL THEOREM FOR GRAVITATIONAL ORBITS

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problems 2.9, 2.10.

We've already met the virial theorem in quantum mechanics where it had the form, in one dimension:

$$2\langle K\rangle=\left\langle x\frac{dU}{dx}\right\rangle \tag{1}$$

where K is the kinetic energy and U is the potential energy. Applying this to an object of mass m in a gravitational orbit about another mass M,

$$U=-\frac{GMm}{r} \tag{2}$$

so using r as our one position dimension, we get

$$2\left\langle K\right\rangle=\left\langle r\frac{dU}{dr}\right\rangle=GMm\left\langle\frac{1}{r}\right\rangle=-\left\langle U\right\rangle \tag{3}$$

For those interested in a completely classical derivation of this result, full details are given in Carroll & Ostlie, chapter 2.

To use this formula, we need to find $\langle U\rangle$, which requires us to average the potential energy over time. Since the motion is periodic, we can find the average over a single orbit, that is, over the time interval $t\in[0,P]$. We have

$$\langle U\rangle=-\frac{GMm}{P}\int_0^P\frac{dt}{r} \tag{4}$$

For an elliptical orbit

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{5}$$

so we can transform the integral to

$$\begin{aligned}
\langle U\rangle &= -\frac{GMm}{Pa\left(1-e^{2}\right)}\int_{0}^{P}\left(1+e\cos\theta\right)dt &(6)\\
&= -\frac{GMm}{Pa\left(1-e^{2}\right)}\int_{0}^{2\pi}\left(1+e\cos\theta\right)\frac{dt}{d\theta}d\theta &(7)\\
&= -\frac{GMm}{Pa\left(1-e^{2}\right)}\int_{0}^{2\pi}\frac{1+e\cos\theta}{\dot{\theta}}d\theta &(8)
\end{aligned}$$

We've already worked out $\dot{\theta}$ as

$$\dot{\theta}=\frac{2\pi\left(1+e\cos\theta\right)^{2}}{P\left(1-e^{2}\right)^{3/2}} \tag{9}$$

so we have to evaluate the integral

$$\begin{aligned}
\int_{0}^{2\pi}\frac{1+e\cos\theta}{\dot{\theta}}d\theta &= \int_{0}^{2\pi}\frac{1+e\cos\theta}{2\pi\left(1+e\cos\theta\right)^{2}}P\left(1-e^{2}\right)^{3/2}d\theta &(10)\\
&= \frac{P\left(1-e^{2}\right)^{3/2}}{2\pi}\int_{0}^{2\pi}\frac{d\theta}{1+e\cos\theta} &(11)
\end{aligned}$$

Carroll & Ostlie give the answer for the integral, but for those interested, the indefinite integral is (using Maple):

$$\int\frac{d\theta}{1+e\cos\theta}=\frac{2}{\sqrt{1-e^{2}}}\arctan\left(\frac{\left(1-e\right)\tan\left(\theta/2\right)}{\sqrt{1-e^{2}}}\right) \tag{12}$$

Evaluating the limits we get

$$\int_{0}^{2\pi}\frac{d\theta}{1+e\cos\theta}=\frac{2\pi}{\sqrt{1-e^{2}}} \tag{13}$$

so

$$\begin{aligned}
\int_{0}^{2\pi}\frac{1+e\cos\theta}{\dot{\theta}}d\theta &= P\left(1-e^{2}\right) &(14)\\
\left\langle\frac{1}{r}\right\rangle &= \frac{1}{a} &(15)\\
\langle U\rangle &= -\frac{GMm}{a} &(16)
\end{aligned}$$

Curiously, although $\left\langle\frac{1}{r}\right\rangle=\frac{1}{a}$, it turns out that $\langle r\rangle\neq a$. Using the same technique as above, we have

$$\langle r\rangle = \frac{a\left(1-e^{2}\right)}{P}\int_{0}^{P}\frac{dt}{1+e\cos\theta} \tag{17}$$

$$= \frac{a\left(1-e^{2}\right)^{5/2}}{2\pi}\int_{0}^{2\pi}\frac{d\theta}{\left(1+e\cos\theta\right)^{3}} \tag{18}$$

The integral can be evaluated in Maple (by imposing the constraint $0<e<1$; the indefinite integral is horrible so I won't give it here):

$$\int_{0}^{2\pi}\frac{d\theta}{\left(1+e\cos\theta\right)^{3}}=\frac{\pi\left(2+e^{2}\right)}{\left(1-e^{2}\right)^{5/2}} \tag{19}$$

so we get

$$\langle r\rangle=\frac{a}{2}\left(2+e^{2}\right) \tag{20}$$

For a circle $e=0$ and $\langle r\rangle=a$, but for any ellipse with a non-zero eccentricity, $\langle r\rangle>a$. Given Kepler's second law (equal areas in equal times), this makes sense in that, since the planet moves more slowly when r is larger, it spends more time at greater distances. However, this doesn't explain why $\left\langle\frac{1}{r}\right\rangle=\frac{1}{a}$; using this logic you'd expect $\left\langle\frac{1}{r}\right\rangle<\frac{1}{a}$.

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