VIRIAL THEOREM FOR GRAVITATIONAL ORBITS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problems 2.9, 2.10.

We've already met the virial theorem in quantum mechanics where it had the form, in one dimension:

$$2\langle K\rangle=\left\langle x\frac{dU}{dx}\right\rangle \tag{1}$$

where K is the kinetic energy and U is the potential energy. Applying this to an object of mass m in a gravitational orbit about another mass M,

$$U=-\frac{GMm}{r} \tag{2}$$

so using r as our one position dimension, we get

$$2\left\langle K\right\rangle=\left\langle r\frac{dU}{dr}\right\rangle=GMm\left\langle\frac{1}{r}\right\rangle=-\left\langle U\right\rangle \tag{3}$$

For those interested in a completely classical derivation of this result, full details are given in Carroll & Ostlie, chapter 2.

To use this formula, we need to find $\langle U\rangle$, which requires us to average the potential energy over time. Since the motion is periodic, we can find the average over a single orbit, that is, over the time interval $t\in[0,P]$. We have

$$\langle U\rangle=-\frac{GMm}{P}\int_0^P\frac{dt}{r} \tag{4}$$

For an elliptical orbit

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{5}$$

so we can transform the integral to

$$\langle U\rangle = -\frac{GMm}{Pa(1-e^2)}\int_0^P (1+e\cos\theta)\,dt \tag{6}$$

$$= -\frac{GMm}{Pa(1-e^2)}\int_0^{2\pi} (1+e\cos\theta)\frac{dt}{d\theta}d\theta \tag{7}$$

$$= -\frac{GMm}{Pa(1-e^2)}\int_0^{2\pi} \frac{1+e\cos\theta}{\dot\theta}d\theta \tag{8}$$

We've already worked out $\dot\theta$ as

$$\dot\theta = \frac{2\pi(1+e\cos\theta)^2}{P(1-e^2)^{3/2}} \tag{9}$$

so we have to evaluate the integral

$$\int_0^{2\pi}\frac{1+e\cos\theta}{\dot\theta}d\theta = \int_0^{2\pi}\frac{1+e\cos\theta}{2\pi(1+e\cos\theta)^2}P\left(1-e^2\right)^{3/2}d\theta \tag{10}$$

$$= \frac{P\left(1-e^2\right)^{3/2}}{2\pi}\int_0^{2\pi}\frac{d\theta}{1+e\cos\theta} \tag{11}$$

Carroll & Ostlie give the answer for the integral, but for those interested, the indefinite integral is (using Maple):

$$\int\frac{d\theta}{1+e\cos\theta} = \frac{2}{\sqrt{1-e^2}}\arctan\left(\frac{(1-e)\tan(\theta/2)}{\sqrt{1-e^2}}\right) \tag{12}$$

Evaluating the limits we get

$$\int_0^{2\pi}\frac{d\theta}{1+e\cos\theta} = \frac{2\pi}{\sqrt{1-e^2}} \tag{13}$$

so

$$\int_0^{2\pi}\frac{1+e\cos\theta}{\dot\theta}d\theta = P\left(1-e^2\right) \tag{14}$$

$$\left\langle\frac{1}{r}\right\rangle = \frac{1}{a} \tag{15}$$

$$\langle U\rangle = -\frac{GMm}{a} \tag{16}$$

Curiously, although $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a}$, it turns out that $\langle r \rangle \neq a$. Using the same technique as above, we have

$$\langle r \rangle = \frac{a\left(1-e^2\right)}{P} \int_0^P \frac{dt}{1+e\cos\theta} \tag{17}$$

$$= \frac{a\left(1-e^2\right)^{5/2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{\left(1+e\cos\theta\right)^3} \tag{18}$$

The integral can be evaluated in Maple (by imposing the constraint $0 < e < 1$; the indefinite integral is horrible so I won't give it here):

$$\int_0^{2\pi} \frac{d\theta}{\left(1+e\cos\theta\right)^3} = \frac{\pi\left(2+e^2\right)}{\left(1-e^2\right)^{5/2}} \tag{19}$$

so we get

$$\langle r \rangle = \frac{a}{2}\left(2+e^2\right) \tag{20}$$

For a circle $e = 0$ and $\langle r \rangle = a$, but for any ellipse with a non-zero eccentricity, $\langle r \rangle > a$. Given Kepler's second law (equal areas in equal times), this makes sense in that, since the planet moves more slowly when r is larger, it spends more time at greater distances. However, this doesn't explain why $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a}$; using this logic you'd expect $\left\langle \frac{1}{r} \right\rangle < \frac{1}{a}$.

PINGBACKS

Pingback: Energy levels of hydrogen: Bohr's semi-classical derivation