

VIRIAL THEOREM FOR GRAVITATIONAL ORBITS

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problems 2.9, 2.10.

We've already met the virial theorem in quantum mechanics where it had the form, in one dimension:

$$(0.1) \quad 2\langle K \rangle = \left\langle x \frac{dU}{dx} \right\rangle$$

where K is the kinetic energy and U is the potential energy. Applying this to an object of mass m in a gravitational orbit about another mass M ,

$$(0.2) \quad U = -\frac{GMm}{r}$$

so using r as our one position dimension, we get

$$(0.3) \quad 2\langle K \rangle = \left\langle r \frac{dU}{dr} \right\rangle = GMm \left\langle \frac{1}{r} \right\rangle = -\langle U \rangle$$

For those interested in a completely classical derivation of this result, full details are given in Carroll & Ostlie, chapter 2.

To use this formula, we need to find $\langle U \rangle$, which requires us to average the potential energy over time. Since the motion is periodic, we can find the average over a single orbit, that is, over the time interval $t \in [0, P]$. We have

$$(0.4) \quad \langle U \rangle = -\frac{GMm}{P} \int_0^P \frac{dt}{r}$$

For an elliptical orbit

$$(0.5) \quad r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

so we can transform the integral to

$$(0.6) \quad \langle U \rangle = -\frac{GMm}{Pa(1-e^2)} \int_0^P (1 + e \cos \theta) dt$$

$$(0.7) \quad = -\frac{GMm}{Pa(1-e^2)} \int_0^{2\pi} (1 + e \cos \theta) \frac{dt}{d\theta} d\theta$$

$$(0.8) \quad = -\frac{GMm}{Pa(1-e^2)} \int_0^{2\pi} \frac{1 + e \cos \theta}{\dot{\theta}} d\theta$$

We've already worked out $\dot{\theta}$ as

$$(0.9) \quad \dot{\theta} = \frac{2\pi(1 + e \cos \theta)^2}{P(1-e^2)^{3/2}}$$

so we have to evaluate the integral

$$(0.10) \quad \int_0^{2\pi} \frac{1 + e \cos \theta}{\dot{\theta}} d\theta = \int_0^{2\pi} \frac{1 + e \cos \theta}{2\pi(1 + e \cos \theta)^2} P(1-e^2)^{3/2} d\theta$$

$$(0.11) \quad = \frac{P(1-e^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 + e \cos \theta}$$

Carroll & Ostlie give the answer for the integral, but for those interested, the indefinite integral is (using Maple):

$$(0.12) \quad \int \frac{d\theta}{1 + e \cos \theta} = \frac{2}{\sqrt{1-e^2}} \arctan \left(\frac{(1-e) \tan(\theta/2)}{\sqrt{1-e^2}} \right)$$

Evaluating the limits we get

$$(0.13) \quad \int_0^{2\pi} \frac{d\theta}{1 + e \cos \theta} = \frac{2\pi}{\sqrt{1-e^2}}$$

so

$$(0.14) \quad \int_0^{2\pi} \frac{1 + e \cos \theta}{\dot{\theta}} d\theta = P(1-e^2)$$

$$(0.15) \quad \left\langle \frac{1}{r} \right\rangle = \frac{1}{a}$$

$$(0.16) \quad \langle U \rangle = -\frac{GMm}{a}$$

Curiously, although $\langle \frac{1}{r} \rangle = \frac{1}{a}$, it turns out that $\langle r \rangle \neq a$. Using the same technique as above, we have

$$(0.17) \quad \langle r \rangle = \frac{a(1-e^2)}{P} \int_0^P \frac{dt}{1+e \cos \theta}$$

$$(0.18) \quad = \frac{a(1-e^2)^{5/2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1+e \cos \theta)^3}$$

The integral can be evaluated in Maple (by imposing the constraint $0 < e < 1$; the indefinite integral is horrible so I won't give it here):

$$(0.19) \quad \int_0^{2\pi} \frac{d\theta}{(1+e \cos \theta)^3} = \frac{\pi(2+e^2)}{(1-e^2)^{5/2}}$$

so we get

$$(0.20) \quad \langle r \rangle = \frac{a}{2}(2+e^2)$$

For a circle $e = 0$ and $\langle r \rangle = a$, but for any ellipse with a non-zero eccentricity, $\langle r \rangle > a$. Given Kepler's second law (equal areas in equal times), this makes sense in that, since the planet moves more slowly when r is larger, it spends more time at greater distances. However, this doesn't explain why $\langle \frac{1}{r} \rangle = \frac{1}{a}$; using this logic you'd expect $\langle \frac{1}{r} \rangle < \frac{1}{a}$.

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