

ORBITAL VELOCITIES AT PERIHELION AND APHELION

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problem 2.13.

The angular momentum of a two-body system is

$$(1) \quad \mathbf{L} = \mu \mathbf{r} \times \mathbf{v}$$

where μ is the reduced mass. At aphelion and perihelion, $\mathbf{r} \perp \mathbf{v}$ and since \mathbf{L} is conserved at all points in the orbit

$$(2) \quad \mu r_a v_a = \mu r_p v_p$$

From the polar equation for an ellipse

$$(3) \quad r = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$(4) \quad r_a = \frac{a(1-e^2)}{1-e} = a(1+e)$$

$$(5) \quad r_p = \frac{a(1-e^2)}{1+e} = a(1-e)$$

so the ratio of the velocities is

$$(6) \quad \frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1+e}{1-e}$$

The energy of the system is

$$(7) \quad E = \frac{\mu}{2}v^2 - \frac{GM\mu}{r}$$

Since E is conserved, we can equate the energies at aphelion and perihelion to get

$$(8) \quad \frac{v_a^2}{2} - \frac{GM}{r_a} = \frac{v_p^2}{2} - \frac{GM}{r_p}$$

From 6 we get

$$(9) \quad v_p = \frac{1+e}{1-e} v_a$$

$$(10) \quad \left[\left(\frac{1+e}{1-e} \right)^2 - 1 \right] v_a^2 = 2GM \left(\frac{1}{r_p} - \frac{1}{r_a} \right)$$

$$(11) \quad \frac{4e}{(1-e)^2} v_a^2 = 2GM \frac{r_a - r_p}{r_a r_p}$$

$$(12) \quad = \frac{4GMae}{a^2(1-e^2)}$$

$$(13) \quad v_a^2 = \frac{GM}{a} \frac{1-e}{1+e}$$

where we used 4 and 5 to get the fourth line. Substituting 9 into the last line we get

$$(14) \quad v_p^2 = \frac{GM}{a} \frac{1+e}{1-e}$$

From 1 at aphelion, 4 and 13 we get

$$(15) \quad L = \mu r_a v_a = \mu a (1+e) \sqrt{\frac{GM}{a} \frac{1-e}{1+e}} = \mu \sqrt{GMa(1-e^2)}$$