ORBITAL VELOCITIES AT PERIHELION AND APHELION

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 2, Problem 2.13.

The angular momentum of a two-body system is

$$\mathbf{L}=\mu\mathbf{r}\times\mathbf{v} \tag{1}$$

where μ is the reduced mass. At aphelion and perihelion, $\mathbf{r}\perp\mathbf{v}$ and since $\mathbf{L}$ is conserved at all points in the orbit

$$\mu r_a v_a=\mu r_p v_p \tag{2}$$

From the polar equation for an ellipse

$$\begin{aligned}
r &= \frac{a\left(1-e^2\right)}{1+e\cos\theta} &(3)\\
r_a &= \frac{a\left(1-e^2\right)}{1-e}=a\left(1+e\right) &(4)\\
r_p &= \frac{a\left(1-e^2\right)}{1+e}=a\left(1-e\right) &(5)
\end{aligned}$$

so the ratio of the velocities is

$$\frac{v_p}{v_a}=\frac{r_a}{r_p}=\frac{1+e}{1-e} \tag{6}$$

The energy of the system is

$$E=\frac{\mu}{2}v^2-\frac{GM\mu}{r} \tag{7}$$

Since E is conserved, we can equate the energies at aphelion and perihelion to get

$$\frac{v_a^2}{2}-\frac{GM}{r_a}=\frac{v_p^2}{2}-\frac{GM}{r_p} \tag{8}$$

From 6 we get

$$\begin{aligned}
v_p &= \frac{1+e}{1-e}v_a && (9)\\
\left[\left(\frac{1+e}{1-e}\right)^2-1\right]v_a^2 &= 2GM\left(\frac{1}{r_p}-\frac{1}{r_a}\right) && (10)\\
\frac{4e}{(1-e)^2}v_a^2 &= 2GM\frac{r_a-r_p}{r_ar_p} && (11)\\
&= \frac{4GMae}{a^2(1-e^2)} && (12)\\
v_a^2 &= \frac{GM}{a}\frac{1-e}{1+e} && (13)
\end{aligned}$$

where we used 4 and 5 to get the fourth line. Subsituting 9 into the last line we get

$$v_p^2 = \frac{GM}{a}\frac{1+e}{1-e} \tag{14}$$

From 1 at aphelion, 4 and 13 we get

$$L = \mu r_a v_a = \mu a(1+e)\sqrt{\frac{GM}{a}\frac{1-e}{1+e}} = \mu\sqrt{GMa(1-e^2)} \tag{15}$$