BLACKBODY RADIATION IN THE FREQUENCY DOMAIN

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 3, Problem 3.13.

The blackbody radiation rate in terms of wavelength is

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5\left(e^{hc/\lambda k_B T} - 1\right)} \tag{1}$$

B_λ is the rate at which a blackbody at temperature T radiates in watts per unit area, per unit wavelength band, per steradian. The amount of radiation emitted in a wavelength span $[\lambda, \lambda + d\lambda]$ is therefore $B_\lambda d\lambda$. To convert this to a rate per unit frequency interval, we need to convert $B_\lambda d\lambda$ to the equivalent form $B_\nu d\nu$. We have

$$\begin{aligned}
\nu &= \frac{c}{\lambda} && (2)\\
d\nu &= -\frac{c}{\lambda^2} d\lambda && (3)\\
|B_\lambda d\lambda| &= \left|\frac{\lambda^2}{c} B_\lambda d\nu\right| && (4)\\
B_\nu &= \frac{\lambda^2}{c} B_\lambda && (5)\\
&= \frac{2hc}{\lambda^3\left(e^{hc/\lambda k_B T} - 1\right)} && (6)\\
&= \frac{2h\nu^3}{c^2\left(e^{h\nu/k_B T} - 1\right)} && (7)
\end{aligned}$$

The frequency at which B_ν is a maximum is found as usual by solving $dB_\nu/d\nu = 0$. As in the case of finding λ_{max}, this gives rise to an equation that must be solved numerically. We get

$$\frac{dB_\nu}{d\nu} = \frac{6h\nu^2}{c^2\left(e^{h\nu/k_BT}-1\right)} - \frac{2h^2\nu^3 e^{h\nu/k_BT}}{c^2 k_B T\left(e^{h\nu/k_BT}-1\right)^2} = 0 \tag{8}$$

$$3 = e^{h\nu/k_BT}\left(3 - \frac{h\nu}{k_BT}\right) \tag{9}$$

$$3 = e^x(3-x) \tag{10}$$

where $x \equiv h\nu/k_BT$. Solving this using Maple we get $x = 2.821439372$ (or $x = 0$, but that isn't very interesting) and then plugging in $h = 6.62606957 \times 10^{-34}$ and $k_B = 1.3806488 \times 10^{-23}$ (in SI units) we get

$$\nu_{max} = 5.879 \times 10^{10} T \text{ s}^{-1} \tag{11}$$

For the Sun, $T = 5777$ K so $\nu_{max} = 3.396 \times 10^{14}$ s^{-1} which corresponds to a wavelength of

$$\lambda = \frac{c}{\nu_{max}} = 880 \text{ nm} \tag{12}$$

This is in the infrared region. This is different from $\lambda_{max} = 2.901 \times 10^{-3}/T = 502$ nm as calculated from Wien's displacement law because B_λ and B_ν measure different things. B_λ is the radiation per unit wavelength interval and B_ν is the radiation per unit frequency interval and as we've seen from 3, these two intervals are not linearly related. The size of a frequency interval depends on the wavelength (and vice versa).