LORENTZ TRANSFORMATIONS AND CAUSALITY

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.2.

To determine whether two observers can disagree about the temporal order of two events, we can calculate the invariant interval between the events. Using four-vector notation:

$$\Delta s^2 \equiv (\Delta x)_i (\Delta x)^i \tag{1}$$

This gives three possible types of pairs of events:

(1) **Timelike:** If $\Delta s^2 < 0$, then it is possible to find a frame in which the two events occur at the same spatial point, but at different times, since it is the time component $-\left(\Delta x^0\right)^2$ which is negative.

(2) **Lightlike:** If $\Delta s^2 = 0$ then $c^2 (\Delta t)^2 = \Delta x^2$ (if the motion is along the x axis; the argument is similar for arbitrary directions), so the events can be connected by a light signal.

(3) **Spacelike:** If $\Delta s^2 > 0$, then it is possible to find a frame in which the two events occur at the same time but at different places. Different observers may disagree about which event occurs first.

However, causality is also preserved directly from the Lorentz transformations. Using relativistic units where $c = 1$, they are, in 2 dimensions:

$$t' = \gamma (t - \beta x) \tag{2}$$

$$x' = \gamma (x - \beta t) \tag{3}$$

Suppose we observe two events, 1 and 2, in frame S, and that

$$\Delta x \equiv x_2 - x_1 = \alpha (t_2 - t_1) \equiv \alpha \Delta t \tag{4}$$

where α is a positive constant (that is, we're assuming the observer S sees event 2 to the right of event 1, and time t_2 is after t_1). If $\alpha = 1$, then $\Delta x = \Delta t$ and the two events could be connected by a light signal, so event 2 could be caused by event 1. If $\alpha < 1$, then $\Delta x < \Delta t$ so it is possible to travel from x_1 to x_2 at less than the speed of light, which means that event 2 could also be caused by event 1. If $\alpha > 1$ then it is impossible to travel from event 1 to event 2 at less than the speed of light, so the two events cannot be causally connected.

From the Lorentz transformations, we get

$$\Delta t' = \gamma(\Delta t - \beta\Delta x) \tag{5}$$

$$= \gamma\Delta t(1 - \alpha\beta) \tag{6}$$

Since $\beta \leq 1$, if $\alpha \leq 1$ then $\Delta t'$ has the same sign as Δt, so both observers will always agree about the order in which the events occur. In other words, if event 1 can cause event 2, then event 1 must always precede event 2 in all reference frames.

However, if $\alpha > 1$, then if $\frac{1}{\alpha} < \beta \leq 1$, the sign of $\Delta t'$ is opposite to the sign of Δt so that the two observers will disagree about the order of the events. Since the two events are not causally connected in this case, there is no need for the observers to agree on their order.