ANOTHER TRIP TO ALPHA CENTAURI: MORE LORENTZ TRANSFORMATION EXAMPLES

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problems 4.6-4.7.

We're now on another trip to α Centauri to get a few more examples of length contraction and time dilation. For simplicity, we'll take the distance to α Centauri as exactly 4 ly (the actual distance is 4.367 ly). [This analysis is similar to that in which we considered the twin paradox.]

The ship leaves Earth at time $t = t' = 0$, with this event also being the time when the origins of S and S' coincide. The ship's speed in Earth's frame S is $\beta = 0.8$ so the time taken to reach α Centauri in frame S is

$$t = \frac{4}{0.8} = 5 \text{ years} \tag{1}$$

The elapsed time as measured on the ship, in frame S' is

$$t' = \frac{t}{\gamma} = t\sqrt{1-\beta^2} = 5 \times \frac{3}{5} = 3 \text{ years} \tag{2}$$

In this case, only a single clock is needed to measure the time.

The distance to α Centauri as measured in S' is

$$d' = \frac{4}{\gamma} = 2.4 \text{ ly} \tag{3}$$

A radio signal is sent from Earth to the ship every 6 months, as measured by a clock on Earth. In frame S, when the n^{th} signal is sent, the ship is at a distance

$$d_n = 0.8\frac{n}{2} = 0.4n \text{ ly} \tag{4}$$

To find when (as measured by S) the ship receives the signal, let's say that the ship receives the n^{th} signal when it is at a distance D_n from Earth and that this occurs at time T. Then the time taken by the radio signal to reach the ship is just D_n, during which time the ship has moved from d_n to D_n, so

$$D_n = \frac{D_n - d_n}{0.8} \tag{5}$$

$$D_n = 5d_n = 2n \text{ ly} \tag{6}$$

The time interval between successive receptions of the signal by the ship is therefore 2 years plus the 6 months between successive transmissions, or 2.5 years, so the ship will receive 2 transmissions during its journey, with the second transmission arriving just as the ship arrives at α Centauri. The time interval between receptions as measured on the ship is $2.5/\gamma = 1.5$ years.

The ship also sends a radio signal back to Earth every 6 months as measured by the ship. When are these messages received on Earth? To solve this, we can find how far from Earth the ship is when it sends each message. In frame S', the n^{th} message is sent at coordinates $(t'_n, x'_n) = \left(\frac{n}{2}, 0\right)$ so applying an inverse Lorentz transformation to find the ship's coordinates in the S frame:

$$t_n = \gamma\left(t'_n + \beta x'_n\right) \tag{7}$$

$$= \frac{5}{3}\frac{n}{2} = \frac{5}{6}n \tag{8}$$

$$x_n = \gamma\left(x'_n + \beta t'_n\right) \tag{9}$$

$$= \frac{5}{3} \times \frac{4}{5} \times \frac{n}{2} \tag{10}$$

$$= \frac{2}{3}n \tag{11}$$

The time of arrival is the time the pulse was sent (t_n) plus the travel time, which is just x_n:

$$t_{arr} = t_n + x_n = \frac{3}{2}n \text{ years} \tag{12}$$

Thus the radio pulses are received on Earth every 1.5 years (Earth time), which is the same interval as the pulses are received on the ship. This is to be expected, since the two viewpoints are symmetric. [Note that we could have also used this method to calculate the times when the Earth signals are received by the ship.]

Because of the relative motion of source and observer, the radio signals are Doppler shifted. If the source wavelength is $\lambda = 15$ cm, the wavelength received is

$$\lambda_r = \sqrt{\frac{1+\beta}{1-\beta}}\lambda = 3\lambda = 45 \text{ cm} \tag{13}$$

The ship immediately reverses direction and heads back to Earth at $\beta = 0.8$ once it has reached α Centauri, but both the ship and Earth continue to send radio signals at 6 month intervals, as measured by their respective clocks. We can analyze the situation using the second method above. From the ship's frame, it is the Earth that appears to suddenly reverse direction. On the outward journey, the Earth sends signals at $t'_n = 5n/6$ when it is at a distance $x'_n = -2n/3$ (the Earth is to the left of the ship, so x'_n is negative). The ship receives these signals every 1.5 years so it receives one when it is halfway to the star and another just as it reaches the star.

At the time of the turn-around, the ship's frame changes from S' to a new frame S'' with a velocity $-\beta$. If we choose the origin of S'' to coincide with the origins of S and S' then the time and position in S'' when the ship reverses are

$$\begin{aligned}
t'' &= \gamma(t-(-\beta)x) && (14)\\
&= \frac{5}{3}(5+0.8\times 4) && (15)\\
&= \frac{41}{3} \text{ years} && (16)\\
x'' &= \gamma(x+\beta t) && (17)\\
&= \frac{5}{3}(4+0.8\times 5) && (18)\\
&= \frac{40}{3} \text{ ly} && (19)
\end{aligned}$$

The time on Earth, as viewed from the ship, jumps as the ship reverses. Before the reverse, $t' = 3$ so, since on Earth $x = 0$:

$$\begin{aligned}
t' &= \gamma(t-\beta x) && (20)\\
3 &= \frac{5}{3}(t-0) && (21)\\
t &= \frac{9}{5} && (22)
\end{aligned}$$

After the reverse

$$t'' = \gamma(t+\beta x) \tag{23}$$
$$\frac{41}{3} = \frac{5}{3}t \tag{24}$$
$$t = \frac{41}{5} \tag{25}$$

When the ship arrives back on Earth, $t=10$ so t'' is

$$t'' = \gamma(t+\beta x) \tag{26}$$
$$= \frac{5}{3}\times 10 = \frac{50}{3} \tag{27}$$

Thus in the ship's frame, the return journey again takes $\frac{50}{3}-\frac{41}{3}=3$ years.

To analyze the signals sent between Earth and the ship, we can use the technique above. We've already seen that when the ship is outbound, signals are received at intervals of 1.5 years by each end. What happens when the ship is returning?

In the S'' frame, the position and time for the event when Earth sends a signal are:

$$x'' = \gamma(x+\beta t) \tag{28}$$
$$= \frac{5}{3}\times\frac{4}{5}\times\frac{n}{2} \tag{29}$$
$$= \frac{2}{3}n \tag{30}$$
$$t'' = \gamma t \tag{31}$$
$$= \frac{5}{6}n \tag{32}$$

These are the same values as in the S' frame, but the difference here is that in S'' the ship's position is $\frac{40}{3}$ so the distance to Earth is $\frac{40}{3}-\frac{2}{3}n$. Thus the times at which the signals arrive are

$$t''_{arr} = \frac{5}{6}n+\frac{40}{3}-\frac{2}{3}n = \frac{n}{6}+\frac{40}{3} \tag{33}$$

Thus signals now arrive 6 times a year instead of every 1.5 years. In the case of the signals sent from Earth, the first two arrive on the outbound leg and account for 3 years in the frame S'. The remaining 18 signals (there are a total of 20 signals sent over the 10 years of the trip as measured on Earth) arrive at intervals of $\frac{1}{6}$ year so take $\frac{18}{6}=3$ years, giving a round trip of 6 years as measured in S'.

For the signals sent from the ship, there are 6 signals sent on the outbound leg and 6 more on the return trip. The first 6 arrive at intervals of 1.5 years, accounting for $6\times1.5=9$ years. The last 6 arrive at intervals of $\frac{1}{6}$ year, accounting for the final year, giving a total of 10 years as measured in S.

To summarize:

For signals sent from Earth:

Time sent (S)	Time arrived (S')
0.5	$\frac{3}{2}$
1.0	3
1.5	$3\frac{1}{6}$
2.0	$3\frac{2}{6}$
2.5	$3\frac{3}{6}$
3.0	$3\frac{4}{6}$
3.5	$3\frac{5}{6}$
4.0	4
4.5	$4\frac{1}{6}$
5.0	$4\frac{2}{6}$
5.5	$4\frac{3}{6}$
6.0	$4\frac{4}{6}$
6.5	$4\frac{5}{6}$
7.0	5
7.5	$5\frac{1}{6}$
8.0	$5\frac{2}{6}$
8.5	$5\frac{3}{6}$
9.0	$5\frac{4}{6}$
9.5	$5\frac{5}{6}$
10.0	6

For signals sent from the ship:

Time sent (S')	Time arrived (S)
0.5	1.5
1.0	3
1.5	4.5
2.0	6.0
2.5	7.5
3.0	9.0
3.5	$9\frac{1}{6}$
4.0	$9\frac{2}{6}$
4.5	$9\frac{3}{6}$
5.0	$9\frac{4}{6}$
5.5	$9\frac{5}{6}$
6.0	10.0