RELATIVISTIC ENERGY REVISITED

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problems 4.16-19.

Relativistic momentum (in 3 dimensions) is taken as

$$\mathbf{p} = \gamma m \mathbf{v} \tag{1}$$

where $\mathbf{v}$ is the particle's velocity in some frame and

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2}$$

Note that this definition applies to a *single* frame of reference; there are no transformations between frames involved here. The use of the symbol γ in this case can therefore be a bit misleading, as it usually means the factor used in Lorentz transformations between two frames, so be careful in your interpretation.

The 'derivation' of this formula really involves an assumption about the form that momentum must have in relativity. The approach of Griffiths is to assume that 1 is true and then show that momentum is conserved in all inertial frames as a result. Carroll & Ostlie's approach is to start by requiring conservation of momentum, then assuming that $\mathbf{p} = f(v) m\mathbf{v}$ for some function f that depends only on the magnitude $|\mathbf{v}|$, then showing that by requiring $f \to 1$ as $v \to 0$ (thus reducing to the Newtonian value of $\mathbf{p} = m\mathbf{v}$ for slow speeds) that $f = \gamma$ as given above. Either way requires an assumption about the form of $\mathbf{p}$, so there doesn't seem to be any way of deriving it from something more fundamental.

Griffiths then proceeds to derive the equation $E = \gamma mc^2$ by defining the 0 component of the momentum four-vector and invoking its conservation. Carroll & Ostlie use a different approach which is worth spelling out here in more detail.

Suppose we consider motion in one dimension and start with a particle at rest, then apply a constant force to the particle. The work done by the force after some time is equal to the kinetic energy of the particle at that time, so we get

$$
\begin{aligned}
F &= \frac{dp}{dt} && (3)\\
K &= \int_{x_i}^{x_f} F\,dx && (4)\\
&= \int_{x_i}^{x_f} \frac{dp}{dt}\,dx && (5)\\
&= \int_{p_i}^{p_f} \frac{dx}{dt}\,dp && (6)\\
&= \int_{p_i}^{p_f} v\,dp && (7)
\end{aligned}
$$

We can integrate by parts and use the initial condition that $p_i = 0$ (since the particle is at rest at the start):

$$
\begin{aligned}
K &= v_f p_f - \int_0^{v_f} p\,dv && (8)\\
&= \gamma_f m v_f^2 - m\int_0^{v_f} \frac{v\,dv}{\sqrt{1-v^2/c^2}} && (9)\\
&= \gamma_f m v_f^2 + mc^2\left(\sqrt{1-v_f^2/c^2}-1\right) && (10)\\
&= \gamma_f m v_f^2 + mc^2\left(\frac{1-v_f^2/c^2-\sqrt{1-v_f^2/c^2}}{\sqrt{1-v_f^2/c^2}}\right) && (11)\\
&= mc^2\left(\gamma_f-1\right) = mc^2(\gamma-1) && (12)
\end{aligned}
$$

where we've dropped the f subscript at the end since the result applies to a particle moving with velocity v in general.

At this point, an assumption is made that this expression for the kinetic energy is actually the difference between the *total* energy of the particle $E_{tot} = \gamma mc^2$ and an energy $E = mc^2$ that is independent of the velocity; this latter energy is identified as the rest energy. The approach of Griffiths above shows that with this assumption, relativistic energy is conserved, so the assumption makes physical sense (and is, of course, verified in experiments).

Example 1. How fast does a particle have to move to have a kinetic energy equal to its rest energy? This happens when $\gamma = 2$ so

$$\frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}} = \frac{\sqrt{3}}{2} = 0.866 \tag{13}$$

Example 2. If $v \ll c$, then

$$\gamma = 1+\frac{1}{2}\frac{v^2}{c^2}+\mathcal{O}\left(\left(\frac{v}{c}\right)^4\right) \tag{14}$$

$$K \rightarrow \frac{1}{2}mc^2\frac{v^2}{c^2}=\frac{1}{2}mv^2 \tag{15}$$

so we reclaim the Newtonian limit.

Example 3. Using 1 and 12 we can show that $K = p^2/\left(m(1+\gamma)\right)$:

$$\begin{aligned}
\frac{p^2}{(1+\gamma)m} &= \frac{\gamma^2 mv^2}{1+\gamma} && (16)\\
&= \frac{\gamma^2 mc^2}{1+\gamma}\frac{v^2}{c^2} && (17)\\
&= \frac{\gamma^2 mc^2}{1+\gamma}\left(1-\frac{1}{\gamma^2}\right) && (18)\\
&= \frac{\left(\gamma^2-1\right)mc^2}{1+\gamma} && (19)\\
&= (\gamma-1)mc^2 && (20)
\end{aligned}$$

Thus in the low velocity limit, $\gamma \rightarrow 1$ and $K \rightarrow p^2/2m$, which is just another way of writing the Newtonian formula.

Example 4. We can also derive the formula giving the total energy in terms of momentum, with no explicit reference to velocity.

$$\begin{aligned}
E^2 &= \gamma^2 m^2 c^4 && (21)\\
&= m^2c^4\left[\gamma^2\left(1-\frac{1}{\gamma^2}\right)+1\right] && (22)\\
&= m^2c^4\left(\gamma^2\frac{v^2}{c^2}+1\right) && (23)\\
&= \gamma^2 m^2 v^2 c^2 + m^2 c^4 && (24)\\
&= p^2c^2+m^2c^4 && (25)
\end{aligned}$$

This formula applies to all particles, even those such as photons with zero mass, provided we can get a value for their momentum. This is another assumption, since if we set $m = 0$ in the first line, we just get $E = 0$, so there is a bit of a kludge going on here. However, again, the equation is verified by experiment.

PINGBACKS

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