ENERGY LEVELS OF HYDROGEN: BOHR'S SEMI-CLASSICAL DERIVATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 5, Problem 5.8.

We've seen how to derive the Bohr formula for the energy levels in hydrogen by solving the Schrödinger equation, but this isn't the way Bohr originally derived the formula; in fact, he couldn't have done it that way since he arrived at the formula before Schrödinger came up with his equation.

Although Bohr's derivation isn't correct, as it relies a lot on classical physics, it's interesting to see how he did it. His idea was to take the electron as a classical particle in a circular orbit around the proton, and equate the Coulomb force of attraction with the centripetal force required to keep the electron in its orbit. In the centre of mass frame, this gives

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{\mu v^2}{r} \tag{1}$$

where e is the proton charge, $\mu = m_p m_e/(m_p + m_e)$ is the reduced mass, v is the velocity of the electron in its orbit and r is the separation of the electron and proton. The kinetic energy is therefore

$$K = \frac{\mu v^2}{2} = \frac{e^2}{8\pi\epsilon_0 r} = -\frac{U}{2} \tag{2}$$

where U is the potential energy (this agrees with the virial theorem, where the binding force is electrostatic rather than gravitational). Therefore the total energy of the hydrogen atom is

$$E = K + U = \frac{U}{2} = -\frac{e^2}{8\pi\epsilon_0 r} \tag{3}$$

This is entirely classical physics, but it is at this point that Bohr introduced the quantum assumption. He proposed that the angular momentum of the system is quantized in units of Planck's constant, so that the only allowed values are

$$L = \mu r v = n\hbar \tag{4}$$

for some positive integer n. From 1, this gives

$$\frac{e^2}{4\pi\epsilon_0} = \frac{L^2}{\mu r} = \frac{n^2\hbar^2}{\mu r} \tag{5}$$

$$E = -\frac{e^2}{8\pi\epsilon_0 r} \tag{6}$$

$$= -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{\mu}{2n^2\hbar^2} \tag{7}$$

$$= -\frac{13.6 \text{ eV}}{n^2} \tag{8}$$

As our quantum mechanical derivation assumed a stationary proton (and since $m_e \ll m_p$), $\mu \approx m_e$ and we get the Bohr formula for the energy levels of hydrogen. Although this derivation is obviously a lot simpler than the series method we had to use to solve the Schrödinger equation, it's also obviously not correct, as it assumes that the electron is a solid particle in a fixed orbit about the proton. However, it seems a bit too much of a coincidence that the semi-classical derivation gives the right answer.

The derivation is easily generalized to an electron orbiting a nucleus containing Z protons, since we just replace e^2 by Ze^2 to get

$$E_Z = -\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2 \frac{\mu}{2n^2\hbar^2} \tag{9}$$

$$= -13.6 \text{ eV}\frac{Z^2}{n^2} \tag{10}$$

The radius of the orbit in the nth quantum state (assuming the classical model) is, from 5

$$r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{\mu Z e^2} \tag{11}$$

With $Z = n = 1$, this gives the Bohr radius:

$$a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} = 5.29177 \times 10^{-11} \text{ m} \tag{12}$$

For ionized helium, $Z = 2$ and the ground state radius is

$$r = \frac{a}{2} = 2.645885 \times 10^{-11} \text{ m} \tag{13}$$

The ground state energy is

$$E = 4(-13.6\text{ eV}) = -54.4\text{ eV} \tag{14}$$

For doubly ionized lithium, $Z = 3$ so

$$r = \frac{a}{3} = 1.763923 \times 10^{-11}\text{ m} \tag{15}$$

$$E = 9(-13.6\text{ eV}) = -122.4\text{ eV} \tag{16}$$

PINGBACKS

Pingback: Gravitational hydrogen atom
Pingback: Hydrogen spectrum: Lyman, Balmer and Paschen series
Pingback: De Broglie waves
Pingback: Saha equation breaks down at the centre of the Sun