MAXWELL-BOLTZMANN VELOCITY DISTRIBUTION

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 8, Problems 8.1 - 8.4.

The spectrum of a star results from photons of particular wavelengths being absorbed or emitted as electrons change their energy levels in the atoms of the star's atmosphere. An absorption (dark) line in the spectrum results when photons are absorbed by electrons to increase their energy level; emission (bright) lines are caused by electrons dropping down in energy levels and emitting the energy difference as photons.

To work out the relative strengths of the various spectral lines, we need to know how many atoms are in each energy state to begin with. For example, if we consider a star composed entirely of hydrogen, if almost all of the hydrogen is in the ground state, we would expect absorption lines corresponding to energy transitions from the ground state to higher states to be dominant.

The emission and absorption of photons is not the only means by which atoms can exchange energy, however. Since a star's atmosphere is at a high temperature, the atoms are moving very quickly. Collisions between atoms are common, and in each collision, energy can be transmitted from one atom to another. The starting point for determining a star's spectrum is therefore the Maxwell-Boltzmann velocity distribution function, which applies to an ideal gas in thermal equilibrium (a reasonable approximation for a stellar atmosphere). This is a result from statistical mechanics which we won't derive here. The number dn of molecules per unit volume with a velocity between v and $v+dv$ is

$$dn=n\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-mv^{2}/2kT}4\pi v^{2}dv \tag{1}$$

Here, n is the number density of gas molecules, each of mass m, k is Boltzmann's constant and T is the temperature.

The quantity kT is a common measure of the thermal energy associated with a given temperature. In a gas, each translational and rotational degree of freedom of a molecule has an energy of $\frac{1}{2}kT$. Since Boltzmann's constant in SI units is very small,

$$k = 1.38064852 \times 10^{-23} \text{J K}^{-1} \tag{2}$$

it's more common to express it in electron volts (eV) per kelvin. Since one joule is

$$1 \text{ J} = 6.241509 \times 10^{18} \text{ eV} \tag{3}$$

Boltzmann's constant is equivalent to

$$k = 8.61733 \times 10^{-5} \text{eV K}^{-1} \tag{4}$$

At room temperature ($T = 300$ K), the thermal energy is

$$kT = 0.0258 \text{ eV} \approx \frac{1}{40} \text{ eV} \tag{5}$$

For $kT = 1$ eV, we have

$$T = \frac{1}{k} = 11,600 \text{ K} \tag{6}$$

This is a high temperature, even for a stellar atmosphere.

For $kT = 13.6$ eV (the ionization energy of hydrogen), we have

$$T = \frac{13.6}{k} = 157,800 \text{ K} \tag{7}$$

which is well above the surface temperature of even the hottest stars.

The most probable speed v_{mp} at a given temperature can be found by taking the derivative of 1 with respect to v and setting to zero. We get

$$\frac{\partial n}{\partial v} = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{v}{kT} \left(2kT - mv^2\right) = 0 \tag{8}$$

The positive, non-zero root is

$$v_{mp} = \sqrt{\frac{2kT}{m}} \tag{9}$$

Using Maple, or integral tables, we can also work out the root-mean-square velocity

$$v_{rms} = \sqrt{\overline{v^2}} \tag{10}$$

$$= \sqrt{\frac{1}{n}\int v^2 dn} \tag{11}$$

$$= \left[4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty e^{-mv^2/2kT}v^4 dv\right]^{1/2} \tag{12}$$

$$= \sqrt{\frac{3kT}{m}} \tag{13}$$

For relatively narrow ranges of velocity, we can use 1 to estimate what fraction of atoms are within a given range of velocities. From Carroll & Ostlie's figure 8.6, for hydrogen at $T = 10,000$ K, $v_{mp} = 1.29 \times 10^4$m s^{-1}. The fraction of molecules with speeds within $\pm 10^3$ m s^{-1} of the most probable speed is then approximately (this gives an upper limit, since the curve peaks at v_{mp}):

$$\Delta n \approx \frac{1}{n} n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv_{mp}^2/2kT} 4\pi v_{mp}^2 \left(2\times 10^3\right) \tag{14}$$

The values are

$$m = 1.6726219 \times 10^{-27}\text{kg} \tag{15}$$

$$k = 1.38064852 \times 10^{-23}\text{J K}^{-1} \tag{16}$$

$$T = 10^4\text{K} \tag{17}$$

$$v_{mp} = 1.29 \times 10^4\text{m s}^{-1} \tag{18}$$

We get

$$\Delta n \approx 0.129 \tag{19}$$

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