SAHA EQUATION FOR STELLAR ATMOSPHERES

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 8, Problems 8.7 - 8.8.

The atoms in a stellar atmosphere are found in various energy states, with the ratios between pairs of states given by the Boltzmann equation

$$\frac{N_b}{N_a}=\frac{g_b}{g_a}e^{-(E_b-E_a)/kT} \tag{1}$$

where $g_{a,b}$ are the degeneracies of states a and b, that is, the number of different quantum states that have the same energy.

This equation assumes, however, that the ionization stage of the atom never changes and that, as temperature is increased, the atom merely jumps to higher and higher energy states while still hanging on to its electrons.

In reality, due to collisions with other atoms or photons, an atom can lose one or more electrons entirely and make a transition to an ionization stage. Thus to get a complete picture of the various atomic states in a star's atmosphere, we need to determine first the distribution of the various ionization stages, and then apply the Boltzmann equation to each separate ionization stage to determine the proportions of atoms within that stage that are at the various energy levels of that ionization stage.

The equation that is used to determine the ratio of one ionization stage to the next is the *Saha equation*. To understand the Saha equation, we need a bit of notation.

The number N_i is the number of atoms in ionization stage i, where i is written as a Roman numeral to distinguish it from the N_a notation in the Boltzmann equation, which refers to the number of atoms with a specific energy level. The lowest ionization stage is state I, which is a neutral atom (that is, not ionized at all). As each successive electron is removed from the atom, i increases by 1, so N_{II} is an atom with one electron removed, and so on. [The notation can be a bit confusing, since it's tempting to interpret the Roman suffix as the number of electrons removed; just remember that it is actually one more than the number of electrons removed.]

The quantity χ_i is the energy required to ionize an atom in the ground state of ionization stage i. Just as a neutral atom has a sequence of energy states ranging from the ground state up through the various excited states, so too does an ion. Thus χ_i is the energy required to take an ion in the ground state of stage i to the ground state of stage $i+1$.

Finally, we need a quantity from statistical mechanics called the *partition function* which is defined as

$$Z \equiv \sum_{j=1}^{\infty} g_j e^{-\left(E_j - E_1\right)/kT} \tag{2}$$

A partition function applies to a single ionization stage, so each stage i has its own partition function Z_i. Also note that the exponent contains the energy difference between excitation state j and the energy of the ground state for that ionization stage, E_1. g_j as usual is the degeneracy of energy state j. Note that the partition function is dimensionless.

With these definitions, the Saha equation is

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \tag{3}$$

where n_e is the number density (per unit volume) of free electrons (that is, electrons that are not attached to any atom) and m_e is the electron mass.

[As a check at this point, we can work out the dimensions of the quantity on the RHS. kT has the dimensions of energy and h the dimensions of energy $\times$ time $=$ mass $\times$ length^2 $\times$ time^{-1}. Thus the quantity in parentheses has dimensions of mass $\times$ energy/energy^2 $\times$ time^2 = mass/energy $\times$ time^2 $=$ length^{-2}. Raising this to the $\frac{3}{2}$ power gives dimensions of volume^{-1} which cancels the volume units of n_e giving an overall dimensionless result, as it has to be.]

An alternative form of the Saha equation can be written using the ideal gas law for the free electrons

$$P_e = n_e kT \tag{4}$$

Then we get

$$\frac{N_{i+1}}{N_i} = \frac{2kTZ_{i+1}}{P_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \tag{5}$$

The partition function 2 has one disturbing feature. Suppose we want the partition function for neutral hydrogen at a very high temperature. The degeneracy of hydrogen is $g_n = 2n^2$, and for large T, the exponential tends to 1 (since the quantity $E_j - E_1$ lies entirely within the range 0 to 13.6 eV). Thus the partition function sum diverges, which doesn't seem terribly useful.

It seems that the resolution of this problem lies in the fact that in any real gas, as we consider higher energy states, the electron in the atom is more likely to ionize than it is to retain a connection to its parent atom, and that due to interactions between atoms, the ionization threshold is lowered a bit so that it requires a bit less than 13.6 eV. Thus the highest energy levels of a neutral hydrogen atom effectively get cut off, since the atom ionizes rather than adopt one of these higher energy states, and the sum in the partition function becomes finite.

In their example 8.1.4, Carroll & Ostlie use the Saha equation to work out the fraction of hydrogen atoms that are ionized as a function of temperature. For most atoms with more electrons than hydrogen, working out the partition function is a complex affair, but for hydrogen we can make a fairly accurate estimate. For neutral hydrogen, $E_1 = -13.6$ eV, $E_2 = -3.4$ eV and $E_3 = -1.5$ eV, with $g_1 = 2$, $g_2 = 8$ and $g_3 = 18$. For $T = 10^4$ K, $kT = \frac{1}{40}\frac{10^4}{300} = 0.833$ eV so the first 3 terms of the partition function are

$$Z_I = 2 + 8 \times 4.8 \times 10^{-6} + 18 \times 4.9 \times 10^{-7} + \ldots \tag{6}$$

Thus to a good approximation we can take $Z_I = 2$. Since ionized hydrogen is just a proton, there is only one energy level and $Z_{II} = 1$.

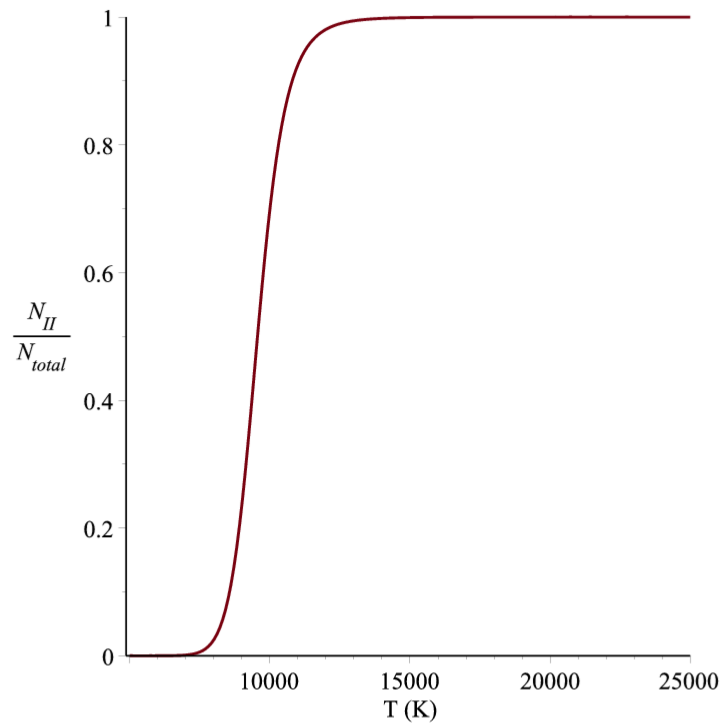
To find the fraction of hydrogen atoms that are ionized, we have

$$\frac{N_{II}}{N_I + N_{II}} = \frac{N_{II}/N_I}{1 + N_{II}/N_I} \tag{7}$$

Plugging in the values in SI units

$$\begin{aligned} k &= 1.38 \times 10^{-23} \text{ J K}^{-1} &(8)\\ m_e &= 9.11 \times 10^{-31} \text{ kg} &(9)\\ h &= 6.63 \times 10^{-34} \text{ J s} &(10)\\ 1 \text{ eV} &= 1.6 \times 10^{-19} \text{ J} &(11) \end{aligned}$$

and using $P_e = 20$ N m^{-2}, we get the following plot for 5000 K $< T <$ 25000 K.



50% ionization occurs at $T = 9558$ K. There is a rapid rise in ionization over a fairly small temperature interval.

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