SAHA EQUATION: NEUTRAL HYDROGEN GAS

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 8, Problem 8.9.

The Saha equation gives the ratio of the number of atoms in ionization stage $i+1$ to those in stage i:

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \tag{0.1}$$

where n_e is the number density (per unit volume) of free electrons (that is, electrons that are not attached to any atom) and m_e is the electron mass. The quantities Z_i and Z_{i+1} are the partition functions of the two ionization stages, and χ_i is the energy required to ionize an atom in the ground state of stage i to the ground state of stage $i+1$.

As an example, suppose we have a volume V of electrically neutral hydrogen gas. In this case, the total number of free electrons $n_e V$ must equal the number of hydrogen ions (free protons) N_{II}

$$n_e V = N_{II} \tag{0.2}$$

If the mass density ρ of the has is known, the total number of hydrogen atoms (both neutral and ionized) is

$$N_t = \frac{\rho V}{m_e + m_p} \approx \frac{\rho V}{m_p} \tag{0.3}$$

where we've neglected the mass of the electron compared to the mass m_p of the proton. We can use these results to determine the fraction N_{II}/N_t of ionized hydrogen in the gas.

For hydrogen, $Z_I = 2$ and $Z_{II} = 1$ so 0.1 can be written as

$$n \equiv \frac{N_{II}}{N_I} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \equiv \frac{A(T)}{n_e} \tag{0.4}$$

where

$$A(T) \equiv \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \tag{0.5}$$

is a function of the temperature but not the number densities.
The ratio that we're after is then

$$\frac{N_{II}}{N_t} = \frac{m_p n_e}{\rho} \tag{0.6}$$

$$= \frac{m_p A(T)}{\rho n} \tag{0.7}$$

We also have

$$\frac{N_{II}}{N_t} = \frac{N_{II}}{N_I + N_{II}} = \frac{n}{1+n} \tag{0.8}$$

$$n = \frac{N_{II}/N_t}{1 - N_{II}/N_t} \tag{0.9}$$

Therefore

$$\frac{N_{II}}{N_t} = \frac{m_p A(T)}{\rho} \frac{1 - N_{II}/N_t}{N_{II}/N_t} \tag{0.10}$$

$$\left(\frac{N_{II}}{N_t}\right)^2 + \frac{m_p A(T)}{\rho} \frac{N_{II}}{N_t} - \frac{m_p A(T)}{\rho} = 0 \tag{0.11}$$

The positive root of this quadratic is

$$\frac{N_{II}}{N_t} = \frac{1}{2}\left(\sqrt{\left(\frac{m_p A}{\rho}\right)^2 + 4\frac{m_p A}{\rho}} - \frac{m_p A}{\rho}\right) \tag{0.12}$$

Plugging in the values in SI units

$$k = 1.38 \times 10^{-23} \text{ J K}^{-1} \tag{0.13}$$

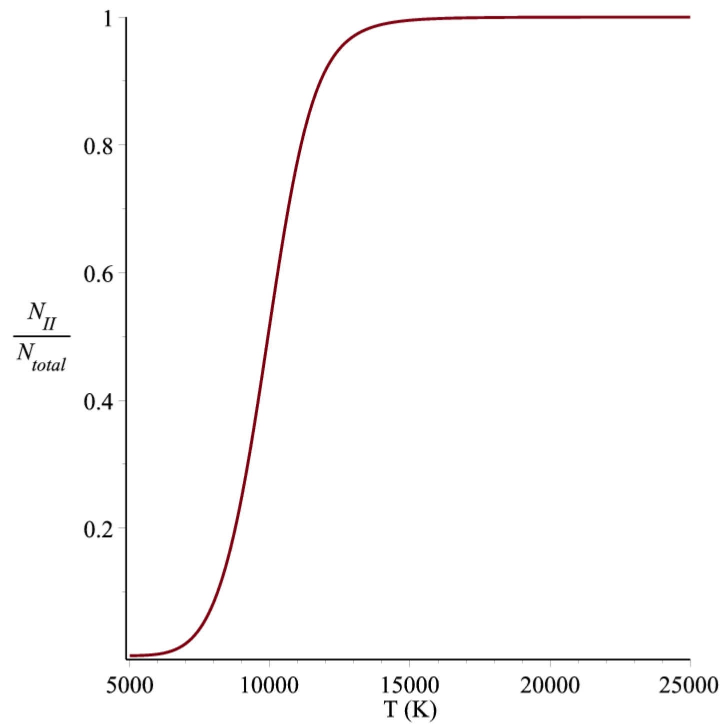
$$m_e = 9.11 \times 10^{-31} \text{ kg} \tag{0.14}$$

$$m_p = 1.67 \times 10^{-27} \text{ kg} \tag{0.15}$$

$$h = 6.63 \times 10^{-34} \text{ J s} \tag{0.16}$$

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \tag{0.17}$$

and using a mass density of $\rho = 10^{-6}$ kg m^{-3} we get the following plot:



Half the atoms are ionized at $T = 9932$ K.