

SAHA EQUATION: NEUTRAL HYDROGEN GAS

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Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 8, Problem 8.9.

The Saha equation gives the ratio of the number of atoms in ionization stage $i + 1$ to those in stage i :

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i/kT} \quad (1)$$

where n_e is the number density (per unit volume) of free electrons (that is, electrons that are not attached to any atom) and m_e is the electron mass. The quantities Z_i and Z_{i+1} are the partition functions of the two ionization stages, and χ_i is the energy required to ionize an atom in the ground state of stage i to the ground state of stage $i + 1$.

As an example, suppose we have a volume V of electrically neutral hydrogen gas. In this case, the total number of free electrons $n_e V$ must equal the number of hydrogen ions (free protons) N_{II}

$$n_e V = N_{II} \quad (2)$$

If the mass density ρ of the gas is known, the total number of hydrogen atoms (both neutral and ionized) is

$$N_t = \frac{\rho V}{m_e + m_p} \approx \frac{\rho V}{m_p} \quad (3)$$

where we've neglected the mass of the electron compared to the mass m_p of the proton. We can use these results to determine the fraction N_{II}/N_t of ionized hydrogen in the gas.

For hydrogen, $Z_I = 2$ and $Z_{II} = 1$ so 1 can be written as

$$n \equiv \frac{N_{II}}{N_I} = \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i/kT} \equiv \frac{A(T)}{n_e} \quad (4)$$

where

$$A(T) \equiv \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i/kT} \quad (5)$$

is a function of the temperature but not the number densities.
The ratio that we're after is then

$$\frac{N_{II}}{N_t} = \frac{m_p n_e}{\rho} \quad (6)$$

$$= \frac{m_p A(T)}{\rho n} \quad (7)$$

We also have

$$\frac{N_{II}}{N_t} = \frac{N_{II}}{N_I + N_{II}} = \frac{n}{1+n} \quad (8)$$

$$n = \frac{N_{II}/N_t}{1 - N_{II}/N_t} \quad (9)$$

Therefore

$$\frac{N_{II}}{N_t} = \frac{m_p A(T)}{\rho} \frac{1 - N_{II}/N_t}{N_{II}/N_t} \quad (10)$$

$$\left(\frac{N_{II}}{N_t}\right)^2 + \frac{m_p A(T)}{\rho} \frac{N_{II}}{N_t} - \frac{m_p A(T)}{\rho} = 0 \quad (11)$$

The positive root of this quadratic is

$$\frac{N_{II}}{N_t} = \frac{1}{2} \left(\sqrt{\left(\frac{m_p A}{\rho}\right)^2 + 4 \frac{m_p A}{\rho}} - \frac{m_p A}{\rho} \right) \quad (12)$$

Plugging in the values in SI units

$$k = 1.38 \times 10^{-23} \text{ J K}^{-1} \quad (13)$$

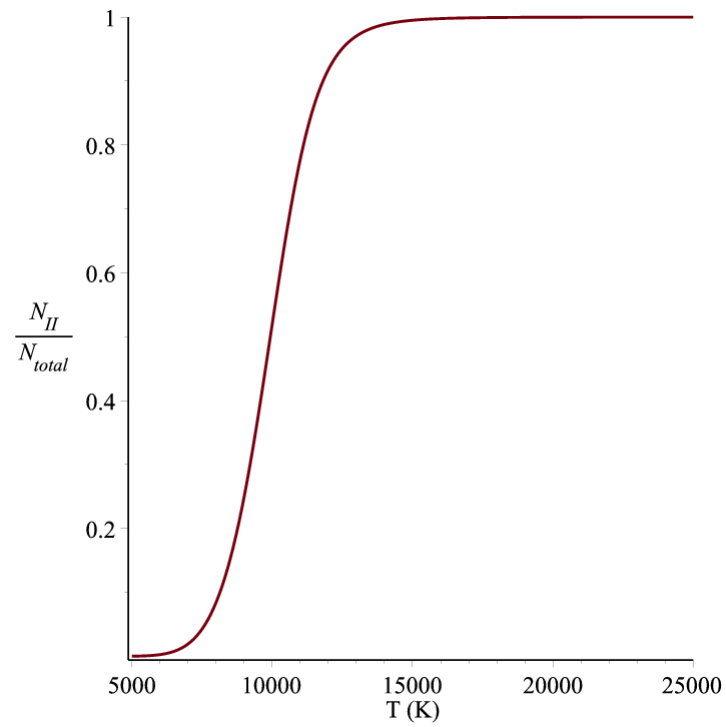
$$m_e = 9.11 \times 10^{-31} \text{ kg} \quad (14)$$

$$m_p = 1.67 \times 10^{-27} \text{ kg} \quad (15)$$

$$h = 6.63 \times 10^{-34} \text{ J s} \quad (16)$$

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \quad (17)$$

and using a mass density of $\rho = 10^{-6} \text{ kg m}^{-3}$ we get the following plot:



Half the atoms are ionized at $T = 9932$ K.