

CYCLIC COORDINATES AND POISSON BRACKETS

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Hamilton's canonical equations are:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad (1)$$

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i \quad (2)$$

1. CYCLIC COORDINATES

If a coordinate q_i is missing in the Hamiltonian (that is, H is independent of q_i), then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \quad (3)$$

Thus the conjugate momentum p_i is conserved. Such a missing coordinate q_i is known as a *cyclic coordinate*. [I'm not sure of the origin of this term. Again Google doesn't provide a definitive answer.]

2. POISSON BRACKETS

There is a general method for calculating the rate of change of some function $\omega(p, q)$ that depends on the momenta and coordinates, but not explicitly on the time (ω is allowed to depend implicitly on time since p and/or q can depend on time). The time derivative can then be written using the chain rule:

$$\begin{aligned} \frac{d\omega}{dt} &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \dot{q}_i + \frac{\partial \omega}{\partial p_i} \dot{p}_i \right) \\ &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &\equiv \{\omega, H\} \end{aligned}$$

where in the second line we used Hamilton's equations 1 and 2. The last line defines the *Poisson bracket* of the function ω with the Hamiltonian H .

$$\boxed{\{\omega, H\} \equiv \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial H}{\partial q_i} \right)} \quad (4)$$

The rate of change of a quantity ω with respect to time is given by its Poisson bracket with the Hamiltonian, provided that H doesn't depend explicitly on time:

$$\boxed{\frac{d\omega}{dt} = \dot{\omega} = \{\omega, H\}} \quad (5)$$

We can see that if $\{\omega, H\} = 0$, the function ω is conserved.

Since $\{H, H\} = 0$ automatically, the total energy (represented by the Hamiltonian) is conserved, provided there is no explicit time dependence. Such a time dependence can arise if the system is subject to some external force, for example.

From the definition 4 we can derive a few fundamental properties of Poisson brackets. We'll consider a general Poisson bracket between two arbitrary functions $\omega(p, q)$ and $\lambda(p, q)$. Then

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \quad (6)$$

$$= - \sum_i \left(\frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} \right) \quad (7)$$

$$= - \sum_i \left(\frac{\partial \lambda}{\partial q_i} \frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \omega}{\partial q_i} \right) \quad (8)$$

$$= - \{\lambda, \omega\} \quad (9)$$

A Poisson bracket is distributive, in the sense that

$$\{\omega, \lambda + \sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda + \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial (\lambda + \sigma)}{\partial q_i} \right) \quad (10)$$

$$= \sum_i \left(\frac{\partial \omega}{\partial q_i} \left[\frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i} \right] - \frac{\partial \omega}{\partial p_i} \left[\frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i} \right] \right) \quad (11)$$

$$= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \quad (12)$$

$$= \{\omega, \lambda\} + \{\omega, \sigma\} \quad (13)$$

One more identity is useful, which we can derive using the product rule:

$$\{\omega, \lambda\sigma\} = \sum_i \left(\frac{\partial\omega}{\partial q_i} \frac{\partial(\lambda\sigma)}{\partial p_i} - \frac{\partial\omega}{\partial p_i} \frac{\partial(\lambda\sigma)}{\partial q_i} \right) \quad (14)$$

$$= \sum_i \sigma \left(\frac{\partial\omega}{\partial q_i} \frac{\partial\lambda}{\partial p_i} - \frac{\partial\omega}{\partial p_i} \frac{\partial\lambda}{\partial q_i} \right) + \sum_i \lambda \left(\frac{\partial\omega}{\partial q_i} \frac{\partial\sigma}{\partial p_i} - \frac{\partial\omega}{\partial p_i} \frac{\partial\sigma}{\partial q_i} \right) \quad (15)$$

$$= \{\omega, \lambda\} \sigma + \{\omega, \sigma\} \lambda \quad (16)$$

The Poisson brackets involving the coordinates q_i and momenta p_i turn up frequently, so it's worth deriving them in detail. We have

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0 \quad (17)$$

This follows because, in the Hamiltonian formalism, the q_i s and p_i s are independent variables, so $\frac{\partial q_j}{\partial p_k} = \frac{\partial p_j}{\partial q_k} = 0$ for all j and k . For the same reason, we have

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \quad (18)$$

The mixed Poisson bracket is a different story, however:

$$\{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \quad (19)$$

$$= \sum_k \delta_{ik} \delta_{jk} - 0 \quad (20)$$

$$= \delta_{ij} \quad (21)$$

Hamilton's equations 1 and 2 can be written using Poisson brackets by setting ω equal to q_i and p_i respectively in 5:

$$\dot{q}_i = \{q_i, H\} \quad (22)$$

$$\dot{p}_i = \{p_i, H\} \quad (23)$$

3. ANGULAR MOMENTUM CONSERVATION

In two dimensions, suppose we have a Hamiltonian:

$$H = p_x^2 + p_y^2 + ax^2 + by^2 \quad (24)$$

If $a = b$, then in polar coordinates, the only coordinate appearing in H is the radial distance from the origin $r = \sqrt{x^2 + y^2}$, which means that the polar

angle θ is a cyclic coordinate. This means that the conjugate momentum p_θ must be conserved. That is,

$$\dot{p}_\theta = \{p_\theta, H\} = 0 \quad (25)$$

However, p_θ is the angular momentum ℓ_z , so this just says that angular momentum is conserved.

To see this explicitly, it's easier to convert to polar coordinates. From Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = 2p_x \quad (26)$$

$$\dot{y} = 2p_y \quad (27)$$

$$p_x^2 + p_y^2 = \frac{1}{4}(x^2 + y^2) \quad (28)$$

$$= \frac{v^2}{4} \quad (29)$$

$$= \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) \quad (30)$$

where in the fourth line, v is the linear velocity and in the fifth line we converted this to polar coordinates. Thus the Hamiltonian becomes, in the case where $a = b$:

$$H = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) + ar^2 \quad (31)$$

To find the conjugate momenta in polar coordinates, we can write out the Lagrangian. We use $p_x\dot{x} = \frac{\dot{x}^2}{2}$ and $p_y\dot{y} = \frac{\dot{y}^2}{2}$ and get

$$L = \sum_i p_i\dot{q}_i - H \quad (32)$$

$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (33)$$

$$= \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (34)$$

The conjugate momenta are thus

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}r^2\dot{\theta} \quad (35)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{2} \quad (36)$$

From this we can see that p_θ is indeed angular momentum as it's proportional to the product of r and the tangential velocity $v_\theta = r\dot{\theta}$. ('Real' momentum and angular momentum must, of course, also contain a factor of a mass, but from the definition of the Hamiltonian above, we see that the mass has been incorporated into the momentum parameters.)

Plugging these back into 31 we get

$$H = p_r^2 + p_\theta^2 + ar^2 \quad (37)$$

We can now calculate the Poisson brackets:

$$\{p_\theta, H\} = \sum_i \left(\frac{\partial p_\theta}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_\theta}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (38)$$

$$= 0 - \frac{\partial p_\theta}{\partial p_\theta} \frac{\partial H}{\partial \theta} = 0 \quad (39)$$

$$\{p_r, H\} = \sum_i \left(\frac{\partial p_r}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_r}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (40)$$

$$= 0 - \frac{\partial p_r}{\partial p_r} \frac{\partial H}{\partial r} \quad (41)$$

$$= -2ar \quad (42)$$

Thus p_θ (the angular momentum) is conserved, while $\dot{p}_r < 0$, meaning that its radial velocity is decreasing.

PINGBACKS

Pingback: Canonical transformations

Pingback: Poisson brackets are invariant under a canonical transformation

Pingback: Passive, regular and active transformations.