

HAMILTONIAN FOR THE TWO-BODY PROBLEM

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Here we derive the equations of motion of the two-body problem using the Hamiltonian formalism.

The Hamiltonian is given by

$$H(q, p) = \sum_i p_i \dot{q}_i - L(q, \dot{q}) \quad (1)$$

where the velocities \dot{q}_i are expressed in terms of the positions q_i and momenta p_i . In this case, we start with the Lagrangian in terms of the centre of mass position \mathbf{r}_{CM} and the relative position \mathbf{r} of mass 2 to mass 1.

$$L = \frac{1}{2}(m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad (2)$$

$$= \frac{M}{2} |\dot{\mathbf{r}}_{CM}|^2 + \frac{\mu}{2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad (3)$$

where $M = m_1 + m_2$ is the total mass and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

There are potentially 6 velocity components and 6 coordinate components in the Lagrangian, but the 3 components of \mathbf{r}_{CM} do not appear, which simplifies things a bit. To convert to a Hamiltonian, we need the momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (4)$$

The x component of momentum of the centre of mass is

$$p_{CM,x} = \frac{\partial L}{\partial \dot{r}_{CM,x}} = M \dot{r}_{CM,x} \quad (5)$$

The other two components of the centre of mass velocity, and of the relative velocity, have a similar form, and in general we can write

$$p_{CM,i} = M \dot{r}_{CM,i} \quad (6)$$

$$p_i = \mu \dot{r}_i \quad (7)$$

In vector notation, this becomes

$$\dot{\mathbf{r}}_{CM} = \frac{\mathbf{p}_{CM}}{M} \quad (8)$$

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{\mu} \quad (9)$$

$$|\dot{\mathbf{r}}_{CM}|^2 = \frac{|\mathbf{p}_{CM}|^2}{M^2} \quad (10)$$

$$|\dot{\mathbf{r}}|^2 = \frac{|\mathbf{p}|^2}{\mu^2} \quad (11)$$

The Lagrangian thus becomes

$$L = \frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r}) \quad (12)$$

The Hamiltonian is

$$H = \mathbf{p} \cdot \dot{\mathbf{r}} + \mathbf{p}_{CM} \cdot \dot{\mathbf{r}}_{CM} - L \quad (13)$$

$$= \frac{|\mathbf{p}|^2}{\mu} + \frac{|\mathbf{p}_{CM}|^2}{M} - \left[\frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r}) \right] \quad (14)$$

$$= \frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + V(\mathbf{r}) \quad (15)$$

Once we've got the Hamiltonian, we can apply Hamilton's canonical equations to get the equations of motion.

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{r}_i \\ \frac{\partial H}{\partial r_i} &= -\dot{p}_i \end{aligned}} \quad (16)$$

Since \mathbf{r}_{CM} does not appear in the Hamiltonian, we have

$$\dot{\mathbf{p}}_{CM} = 0 \quad (17)$$

$$\mathbf{p}_{CM} = \text{constant} \quad (18)$$

so the momentum of the centre of mass does not change, as expected.

For \mathbf{r} , we have

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{\mu} = \dot{r}_i \quad (19)$$

$$\frac{\partial H}{\partial r_i} = \frac{\partial V}{\partial r_i} = -\dot{p}_i \quad (20)$$

The first equation tells us nothing new, while the second is just Newton's law for a central force: $\dot{\mathbf{p}} = -\nabla V$.