

## RELATION BETWEEN ACTION AND ENERGY

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Here we'll examine an interesting relation between the action  $S$  and the total energy of a system, as given by the Hamiltonian  $H$ . Suppose a single particle moving in one dimension follows a classical path given by  $x_{cl}(t)$ , and moves from an initial position at time  $t_i$  of  $x_{cl}(t_i) = x_i$  to a final position at time  $t_f$  of  $x_{cl}(t_f) = x_f$ . The action  $S_{cl}$  of this classical path is given by the integral of the Lagrangian

$$S_{cl} = \int_{t_i}^{t_f} L(x, \dot{x}) dt \quad (1)$$

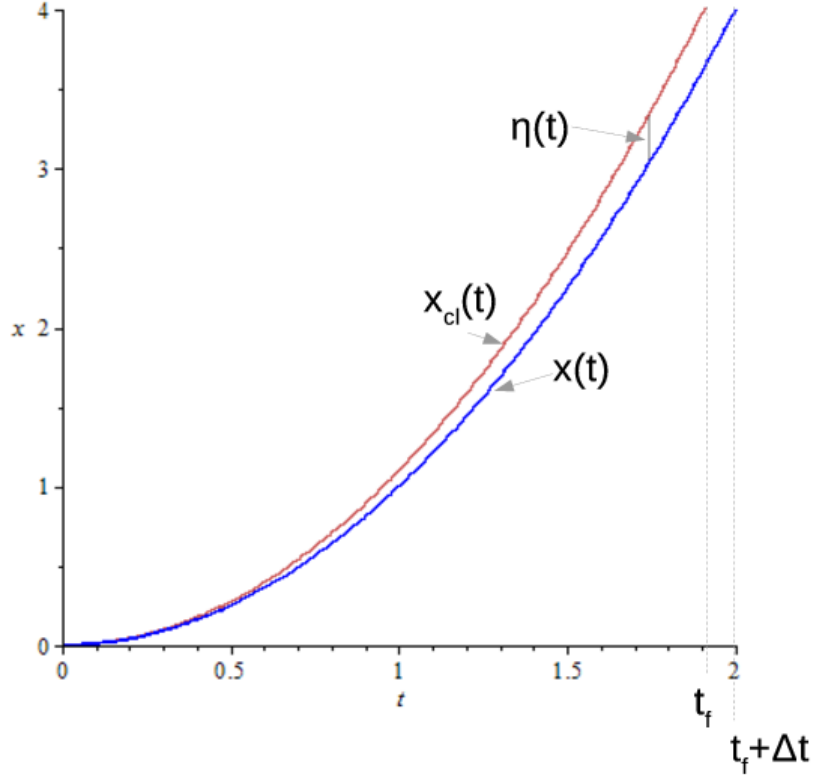
What can we say about the rate of change of the action with respect to the final time  $t_f$ ? That is, we want to calculate  $\partial S_{cl}/\partial t_f$ , where all other parameters  $t_i, x_i$  and  $x_f$  are held constant. The situation can be illustrated as shown in Fig. 1.

Since the only thing that is changing is  $t_f$ , the particle starts at the same initial time (which we've taken to be  $t_i = 0$  in the diagram) and moves to the same location  $x_f$ , but at a different time (in the diagram, later time  $t_f + \Delta t$ ). This means that the particle must follow a different path, possibly over its entire trajectory. This path, which we'll call  $x(t)$ , is related to the original path  $x_{cl}(t)$  by perturbing the original path by an amount  $\eta(t)$ :

$$x(t) = x_{cl}(t) + \eta(t) \quad (2)$$

In Fig. 1, the original path  $x_{cl}$  is shown in red and the perturbed path  $x$  in blue. The amount  $\eta$  is seen to be the vertical distance between these two curves at each time, and in the case of the paths shown in the diagram,  $\eta(t) < 0$  over the entire path, but this isn't necessary. The blue path could be above the red path, or it could take some other path that oscillates around the red path.

The difference in the action  $S$  between the two paths is due to two contributions: first, there is the contribution due to the extra time, from  $t_f$  to

FIGURE 1. Action with variable final time  $t_f$ .

$t_f + \Delta t$ , that the particle takes to complete its path. Second, there is the difference in the two actions over the path from  $t_i$  to  $t_f$ . The first contribution is entirely new and, for an infinitesimal extra time  $\Delta t$ , it is given by

$$\delta S_1 = L(t_f) \Delta t \quad (3)$$

where  $L(t_f)$  is the Lagrangian evaluated at time  $t_f$ . The other contribution can be obtained by varying the action over the path from  $t_i = 0$  to  $t_f$ :

$$\delta S_2 = \int_0^{t_f} \delta L dt \quad (4)$$

Since  $L$  depends on  $x$  and  $\dot{x}$ , we have

$$\delta L = \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \quad (5)$$

For infinitesimally different trajectories, we can see from the diagram above that  $\delta x = \eta(t)$  at each point on the curve, so  $\delta \dot{x} = \dot{\eta}(t)$ , so we get

$$\delta S_2 = \int_0^{t_f} \left[ \frac{\partial L}{\partial x} \eta(t) + \frac{\partial L}{\partial \dot{x}} \dot{\eta}(t) \right] dt \quad (6)$$

$$= \int_0^{t_f} \left[ -\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} \right] \eta(t) dt + \int_0^{t_f} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \eta(t) \right) dt \quad (7)$$

$$= 0 + \left. \frac{\partial L}{\partial \dot{x}} \eta(t) \right|_{t_f} \quad (8)$$

In these equations, the derivatives of  $L$  are evaluated on the original curve  $x_{cl}$ . To verify the second line, use the product rule on the second integrand and cancel terms to get the first line. The second term in the last line is evaluated at  $t = t_f$  only since we're assuming that both paths start at the same point in space and time, so that  $\eta(0) = 0$ .

The quantity in brackets in the first integral is zero, because of the Euler-Lagrange equations which are valid on the original curve  $x_{cl}$ :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad (9)$$

Putting everything together, we get for the total variation in the action:

$$\delta S_{cl} = \delta S_1 + \delta S_2 \quad (10)$$

$$= \left[ \frac{\partial L}{\partial \dot{x}} \eta(t) + L \Delta t \right]_{t_f} \quad (11)$$

Looking at Fig. 1, the slope of the blue curve  $x(t_f)$  at the time  $t_f$  is given by

$$\dot{x}(t_f) = \frac{|\eta(t_f)|}{\Delta t} \quad (12)$$

From the definition 2 of  $\eta$  we see that  $\eta(t_f) < 0$ , so

$$\eta(t_f) = -\dot{x}(t_f) \Delta t \quad (13)$$

This gives the final equation for the variation of the action:

$$\delta S_{cl} = \left[ -\frac{\partial L}{\partial \dot{x}} \dot{x} + L \right]_{t_f} \Delta t \quad (14)$$

$$= (-p\dot{x} + L) \Delta t \quad (15)$$

$$= -H \Delta t \quad (16)$$

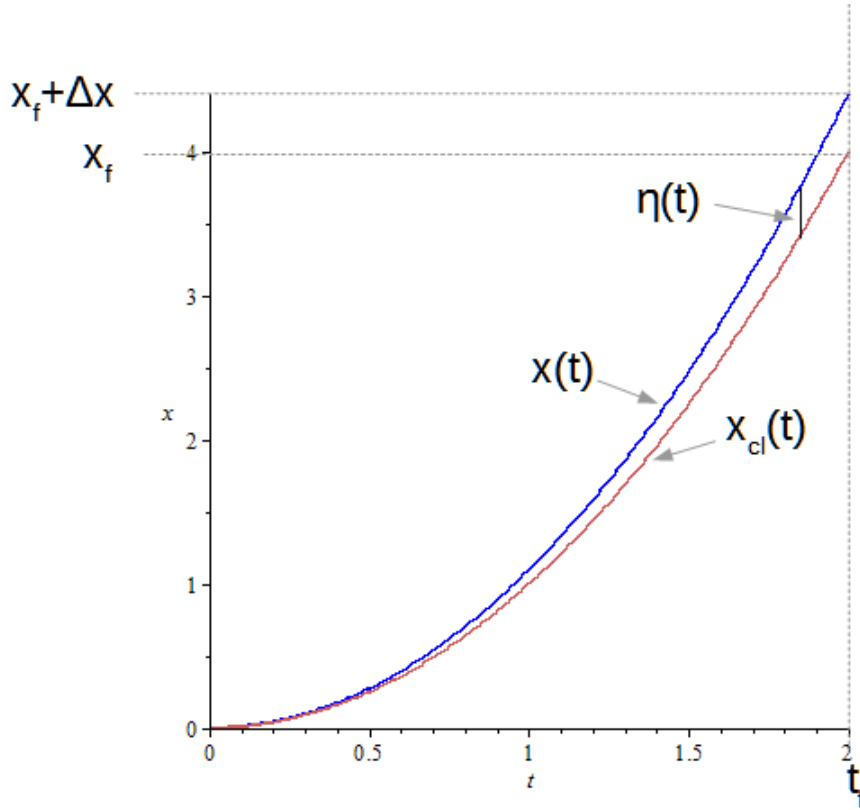


FIGURE 2. Action with variable final position  $x_f$ .

where the second line follows from the definition of the canonical momentum  $p = \partial L / \partial \dot{x}$ .

The required derivative is

$$\boxed{\frac{\partial S_{cl}}{\partial t_f} = -H(t_f)} \quad (17)$$

The derivative of the action with respect to time is the negative of the total energy.

Using a similar technique, we can work out  $\partial S_{cl} / \partial x_f$ . In this case, the situation is as shown in Fig. 2.

The two trajectories now take the same time, but in the modified trajectory, the particle moves a distance  $\Delta x$  further. Since both paths take the same time, there is no extra contribution  $L\Delta t$ . In this case  $\eta(t) > 0$ , since the new (blue) curve  $x(t)$  is above the old (red) one  $x_{cl}(t)$ . The derivation is the same as above up to 8, and the total variation in the action is now

$$\delta S_{cl} = \left. \frac{\partial L}{\partial \dot{x}} \eta(t) \right|_{t_f} \quad (18)$$

At  $t = t_f$ ,  $\eta(t_f) = \Delta x$ , so we get

$$\delta S_{cl} = \left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} \Delta x \quad (19)$$

$$\frac{\partial S_{cl}}{\partial x_f} = \left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} = p(t_f) \quad (20)$$

We can verify 17 for the case of the one-dimensional harmonic oscillator. The general solution for the position is given by

$$x(t) = A \cos \omega t + B \sin \omega t \quad (21)$$

$$\dot{x}(t) = -A\omega \sin \omega t + B\omega \cos \omega t \quad (22)$$

The total energy is given by

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 \quad (23)$$

$$= \frac{m}{2} \left( (-A\omega \sin \omega t + B\omega \cos \omega t)^2 + \omega^2 (A \cos \omega t + B \sin \omega t)^2 \right) \quad (24)$$

$$= \frac{m\omega^2}{2} (A^2 + B^2) \quad (25)$$

where we just multiplied out the second line, cancelled terms and used  $\cos^2 x + \sin^2 x = 1$ .

To get the action, we need the Lagrangian:

$$L = T - V \quad (26)$$

$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \quad (27)$$

$$= \frac{m}{2} \left( (-A\omega \sin \omega t + B\omega \cos \omega t)^2 - \omega^2 (A \cos \omega t + B \sin \omega t)^2 \right) \quad (28)$$

$$= \frac{m\omega^2}{2} [A^2 (\sin^2 \omega t - \cos^2 \omega t) + B^2 (\cos^2 \omega t - \sin^2 \omega t) - 4AB \sin \omega t \cos \omega t] \quad (29)$$

$$= \frac{m\omega^2}{2} ((B^2 - A^2) \cos 2\omega t - 2AB \sin 2\omega t) \quad (30)$$

The action for a trajectory from  $t = 0$  to  $t = T$  is then

$$S = \int_0^T L dt \quad (31)$$

$$= \frac{m\omega}{4} [(B^2 - A^2) \sin 2\omega t + 2AB \cos 2\omega t]_0^T \quad (32)$$

$$= \frac{m\omega}{4} [(B^2 - A^2) \sin 2\omega T + 2AB (\cos 2\omega T - 1)] \quad (33)$$

$$= \frac{m\omega}{2} [(B^2 - A^2) \sin \omega T \cos \omega T + AB (\cos^2 \omega T - \sin^2 \omega T - 1)] \quad (34)$$

$$= \frac{m\omega}{2} [(B^2 - A^2) \sin \omega T \cos \omega T - 2AB \sin^2 \omega T] \quad (35)$$

To proceed further, we need to specify  $A$  and  $B$ , since these depend on the boundary conditions (that is, on where we require the mass to be at  $t = 0$  and  $t = T$ ). If we require  $x(0) = x_1$  and  $x(T) = x_2$ , then

$$A = x_1 \quad (36)$$

$$x_1 \cos \omega T + B \sin \omega T = x_2 \quad (37)$$

$$B = \frac{x_2 - x_1 \cos \omega T}{\sin \omega T} \quad (38)$$

Plugging these into 25 gives the energy as

$$E = \frac{m\omega^2}{2} \left( x_1^2 + \left( \frac{x_2 - x_1 \cos \omega T}{\sin \omega T} \right)^2 \right) \quad (39)$$

$$= \frac{m\omega^2}{2 \sin^2 \omega T} (x_1^2 + x_2^2 - 2x_1 x_2 \cos \omega T) \quad (40)$$

Plugging  $A$  and  $B$  into 35, we get (using  $c \equiv \cos \omega T$  and  $s \equiv \sin \omega T$ , so that  $s^2 + c^2 = 1$ ):

$$S = \frac{m\omega}{2s} [(x_2 - x_1 c)^2 c - x_1 s^2 c - 2x_1 s^2 (x_2 - x_1 c)] \quad (41)$$

$$= \frac{m\omega}{2s} [(x_2^2 - 2x_1 x_2 c + x_1^2 c^2) c - x_1^2 s^2 c - 2x_1 x_2 s^2 + 2x_1 s^2 c] \quad (42)$$

$$= \frac{m\omega}{2s} [(x_1^2 + x_2^2) c - 2x_1 x_2] \quad (43)$$

$$= \frac{m\omega}{2 \sin \omega T} [(x_1^2 + x_2^2) \cos \omega T - 2x_1 x_2] \quad (44)$$

Taking the derivative, we get

$$\frac{\partial S}{\partial T} = \frac{m\omega}{2s^2} [-\omega(x_1^2 + x_2^2)s^2 - ((x_1^2 + x_2^2)c - 2x_1x_2)\omega c] \quad (45)$$

$$= \frac{m\omega^2}{2s^2} [-(x_1^2 + x_2^2) + 2x_1x_2c] \quad (46)$$

$$= -\frac{m\omega^2}{2\sin^2\omega T} (x_1^2 + x_2^2 - 2x_1x_2\cos\omega T) \quad (47)$$

$$= -E \quad (48)$$

For the spatial derivative, we have, starting from 35, and using, from 38 with the final position being  $x_f = x_2$ :

$$\frac{\partial B}{\partial x_f} = \frac{\partial B}{\partial x_2} = \frac{1}{s} \quad (49)$$

we have

$$\frac{\partial S}{\partial x_f} = \frac{\partial S}{\partial x_2} \quad (50)$$

$$= \frac{m\omega}{2} \left[ 2B \frac{\partial B}{\partial x_2} sc - 2A \frac{\partial B}{\partial x_2} s^2 \right] \quad (51)$$

$$= m\omega \left[ \frac{B}{s} sc - A \frac{s^2}{s} \right] \quad (52)$$

$$= m\omega (-A \sin\omega T + B \cos\omega T) \quad (53)$$

From 22, the momentum at time  $T$  is  $p_f = p(x = x_2)$

$$p_f = m\dot{x}(T) \quad (54)$$

$$= m\omega (-A \sin\omega T + B \cos\omega T) \quad (55)$$

Thus both 17 and 20 are verified for the harmonic oscillator.