AMPÈRE'S LAW - SURFACE OF INTEGRATION AND THE STEADY CURRENT ASSUMPTION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 12 Mar 2021.

Ampère's law in integral form states that

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int \mathbf{J}\cdot d\mathbf{a} \tag{1}$$

That is, if we choose a closed loop and integrate $\mathbf{B}\cdot d\mathbf{l}$ around this loop, the result is the same as integrating $\mu_0\mathbf{J}\cdot d\mathbf{a}$ over the area enclosed by this loop. However, a closed loop can be the boundary of an infinite number of surfaces, so since the law doesn't state which surface over which to integrate, does it make any difference which surface we choose?

To analyze this problem, think about what $\mathbf{J}\cdot d\mathbf{a}$ represents. It is the component of current travelling perpendicular to the surface at a given point on the surface. To make things concrete, suppose we take the loop to be a circle, and consider two surfaces bounded by this circle. The first is a flat disk, and the second is a hemisphere.

If the currents are steady (not changing with time), then from conservation of charge, any current that flows through the disk can do one of two things. First, it can continue on and flow through the hemisphere as well or second, it can turn around and flow back out through the disk. It can't enter the region between the disk and the hemisphere and then cycle endlessly within that volume or just pile up somewhere, since if it did, charge would continually be flowing into the volume with no charge flowing out, so the amount of charge within the volume would be increasing, which means that the currents within the volume are not constant, thus violating our assumption of steady currents.

Now if current flowing through the disk continues on to flow through the hemisphere as well, then the amount of current flowing through the two surfaces is the same. On the other hand, if some current that flows through the disk turns around and flows out again without reaching the hemisphere, then that current makes a net contribution of zero to the amount of current flowing through the disk, so again the net amount of current flowing through the two surfaces is the same. Thus the net total amount of current flowing through any surface is the same.

Note in particular that this argument requires the assumption of steady currents, which is expressed mathematically by requiring $\nabla \cdot \mathbf{J} = 0$, which is to say that there are no sources or sinks for current. This assumption was in fact used in the derivation of Ampère's law.

The assumption that $\nabla \cdot \boldsymbol{J} = 0$ makes the differential form of the law consistent. This form states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

An identity from vector calculus states that, for any vector field, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. This is true in the steady-current case, but breaks down in the more general case where $\nabla \cdot \mathbf{J} = \partial\rho/\partial t$, that is, where the current is responsible for redistributing the charge distribution.

There is a related problem with the electric field equation $\nabla \times \mathbf{E} = 0$. This doesn't pose any problems mathematically, since the identity $\nabla \cdot (\nabla \times \mathbf{E}) = 0$ is still true. However, the curl-free field equation was derived from Coulomb's law, which in turn gave rise to the fact that in electrostatics, the field can be written as the gradient of a potential, and the vector identity $\nabla \times \nabla \Phi = 0$ then gives us the zero curl. Once charges start to move, they generate magnetic fields, and as we'll see from Maxwell's equations, varying magnetic fields can affect the electric field and give rise to a non-zero curl.