ANGULAR MOMENTUM IN ELECTROMAGNETIC FIELDS

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The momentum density of an electromagnetic field is given by

$$\mathfrak{p}_{\text{em}} = \epsilon_0\mu_0\mathbf{S} = \epsilon_0\mathbf{E}\times\mathbf{B} \tag{1}$$

If we have linear momentum, then we automatically have *angular momentum* with respect to some origin by using the classical definition of angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. We can define the angular momentum density of an electromagnetic field by

$$\mathfrak{L}_{\text{em}} \equiv \mathbf{r}\times\mathfrak{p}_{\text{em}} = \epsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) \tag{2}$$

Just as with linear momentum, even static fields can have angular momentum. As an example, suppose we have a long solenoid with n turns per unit length carrying current I_{sol} and a radius R, with its axis along the z axis. The magnetic field inside the solenoid is (using cylindrical coordinates)

$$\mathbf{B}_0 = \mu_0 n I_{\text{sol}}\hat{\mathbf{z}} \tag{3}$$

The field is zero outside the solenoid.

Now suppose we add two other cylinders (not solenoids), both coaxial with the solenoid. One cylinder has radius $a<R$ (so it lies inside the solenoid) and carries surface charge $+Q$; the other cylinder has radius $b>R$ (outside the solenoid) and carries charge $-Q$. Both cylinders have length ℓ. From Gauss's law, the electric field between these two cylinders is, for $a<r<b$

$$\mathbf{E}_0 = \frac{Q}{2\pi\epsilon_0\ell}\frac{\hat{\mathbf{r}}}{r} \tag{4}$$

That is, the field points radially outward from the axis. The electric field is zero for $r<a$ and $r>b$. (We're neglecting end effects, so we're assuming that $\ell\gg b>a$.)

The linear momentum density is non-zero in the region $a<r<R$ (where both fields are non-zero) and we have from 1

$$\mathfrak{p}_{\text{em}} = \epsilon_0\boldsymbol{E}_0\times\boldsymbol{B}_0 = -\frac{\mu_0 n I_{\text{sol}}Q}{2\pi\ell}\frac{\hat{\boldsymbol{\phi}}}{r} \tag{5}$$

so the angular momentum density is

$$\begin{aligned}\mathfrak{L}_{\text{em}} &= \mathbf{r}\times\mathfrak{p}_{\text{em}} &(6)\\ &= -\frac{\mu_0 n I_{\text{sol}} Q}{2\pi\ell}\hat{\mathbf{z}} &(7)\end{aligned}$$

Conveniently, this is constant so the total angular momentum is just the density times the volume of the cylindrical tube in the region $a<r<R$

$$\begin{aligned}\mathbf{L}_{\text{em}} &= -\ell\pi\left(R^2-a^2\right)\frac{\mu_0 n I_{\text{sol}} Q}{2\pi\ell}\hat{\mathbf{z}} &(8)\\ &= -\left(R^2-a^2\right)\frac{\mu_0 n I_{\text{sol}} Q}{2}\hat{\mathbf{z}} &(9)\end{aligned}$$

Now suppose we (quasistatically) discharge the two cylinders by connecting a resistor $\mathcal{R}$ between them. We'd like to show that the angular momentum gets transferred from the fields to the physical devices in the problem. The two cylinders are effectively a capacitor with some capacitance C so the voltage and charge are related by

$$V(t)=\frac{Q(t)}{C} \tag{10}$$

From Ohm's law we have

$$V=I\mathcal{R}=\mathcal{R}\frac{dQ}{dt} \tag{11}$$

Therefore

$$\mathcal{R}\frac{dQ}{dt}-\frac{Q}{C}=0 \tag{12}$$

which has the solution

$$\begin{aligned}Q(t) &= Q_0 e^{-t/\mathcal{R}C} &(13)\\ &= V_0 C e^{-t/\mathcal{R}C} &(14)\end{aligned}$$

where V_0 is the potential difference between the cylinders at $t=0$.

This in turn gives us the current

$$I(t)=-\frac{V_0}{\mathcal{R}}e^{-t/\mathcal{R}C} \tag{15}$$

The minus sign indicates that the direction of the current is such that the capacitor discharges. In our case, since the positive cylinder is inside the negative cylinder, the direction of the current is outwards, so it is in the $+\hat{\mathbf{r}}$ direction.

The force $d\mathbf{F}$ on a segment of the resistor of length dr is (I've absorbed the direction of the current into the vector $\hat{\boldsymbol{r}}$, so we deal with the magnitude of the current $|I(t)|$ in what follows.)

$$d\mathbf{F} = |I(t)|\, dr\hat{\mathbf{r}} \times \mathbf{B}_0 \tag{16}$$

$$= -|I(t)|\, dr B_0 \hat{\boldsymbol{\phi}} \tag{17}$$

so the torque on this segment is

$$d\mathbf{N} = \mathbf{r} \times d\mathbf{F} \tag{18}$$

$$= -|I(t)|\, B_0 r\; dr\hat{\mathbf{z}} \tag{19}$$

The total torque on the resistor at time t is

$$\mathbf{N}(t) = -|I(t)|\, B_0 \hat{\mathbf{z}} \int_a^R r\; dr \tag{20}$$

$$= -\frac{1}{2}|I(t)|\, B_0 \left(R^2 - a^2\right)\hat{\mathbf{z}} \tag{21}$$

The angular impulse $\boldsymbol{I}$ is the integral of torque over time, so we get

$$\mathbf{I} = \int_0^\infty \mathbf{N}(t)\, dt \tag{22}$$

$$= -\frac{1}{2}B_0 \left(R^2 - a^2\right)\hat{\mathbf{z}} \int_0^\infty |I(t)|\, dt \tag{23}$$

$$= -\frac{1}{2}B_0 \left(R^2 - a^2\right)\hat{\mathbf{z}} \int_0^\infty \frac{V_0}{\mathcal{R}} e^{-t/\mathcal{R}C} dt \tag{24}$$

$$= -\frac{1}{2}B_0 \left(R^2 - a^2\right) C V_0 \hat{\mathbf{z}} \tag{25}$$

$$= -\frac{1}{2}\mu_0 n I_{\text{sol}} \left(R^2 - a^2\right) Q_0 \hat{\mathbf{z}} \tag{26}$$

where we used the relation between capacitance, charge and voltage $Q = CV$. We see that this agrees with 9, so all the angular momentum in the fields is transferred to the resistor as the electric field is reduced to zero.