AVERAGE FIELD OF A DIPOLE OVER A SPHERE

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Post date: 23 Feb 2021.

We've seen that the potential of a pure dipole with dipole moment $\mathbf{p}$ (that is, a system of charge where only the dipole term in the multipole expansion is non-zero) is, in spherical coordinates:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0 r^2}\mathbf{p}\cdot\hat{\mathbf{r}} &(1)\\ &= \frac{1}{4\pi\epsilon_0 r^2}p\cos\theta &(2)\end{aligned}$$

By taking the gradient in spherical coordinates, we can find the electric field of a dipole, since $\mathbf{E}=-\nabla V$.

$$\begin{aligned}\mathbf{E} &= -\frac{\partial V}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi} &(3)\\ &= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right] &(4)\end{aligned}$$

We can calculate the average field of a dipole at the centre of a sphere of radius R over the sphere, but to do this we need to express the spherical unit vectors in terms of rectangular unit vectors, since the spherical unit vectors chage with position. We have

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}} \tag{5}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}}+\cos\theta\sin\phi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}} \tag{6}$$

Substituting these into the field and collecting terms we get

$$\mathbf{E}=\frac{p}{4\pi\epsilon_0 r^3}[3\sin\theta\cos\theta\cos\phi\hat{\mathbf{x}}+3\sin\theta\cos\theta\sin\phi\hat{\mathbf{y}}+(2\cos^2\theta-\sin^2\theta)\hat{\mathbf{z}}] \tag{7}$$

If we integrate this over the sphere, doing the angular integrals first, we get zero, since (remember we need to multiply by the spherical volume element $r^2\sin\theta\,d\phi\,d\theta\,dr$):

$$\int_0^{2\pi} \cos\phi d\phi = \int_0^{2\pi} \sin\phi d\phi = \int_0^{\pi} (2\cos^2\theta - \sin^2\theta)\sin\theta d\theta = 0 \tag{8}$$

However, the integral over r is infinite, since $\int_0^R (1/r)dr$ is infinite at the lower limit. We are thus left with an 'infinity times zero' situation, so the answer is not well defined.

We've seen that the average field of a charge distribution within a sphere is

$$\mathbf{E}_{av} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} \tag{9}$$

The problem clearly arises at the point $r = 0$, since if we consider integrating the expression for the dipole field above from some infinitesimal sphere $r = \epsilon$ out to $r = R$, we will always get zero due to the angular component. If we accept that the expression for the average field is valid, then we can patch things up by requiring the field to have a delta-function component at $r = 0$. That is, if we say that there is a term

$$\mathbf{E}_0 = -\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}) \tag{10}$$

then the average field comes out right since

$$\frac{3}{4\pi R^3}\int \mathbf{E}_0 d^3\mathbf{r} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} \tag{11}$$

We can't really rewrite the total field as

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}) \tag{12}$$

since the first term applies only at $r \neq 0$. As always with delta functions, the physical interpretation is a bit dodgy. I suspect the problem may lie in the fact that 7 isn't really valid for r near zero. Any true dipole must take up some non-zero physical space so it isn't really correct to assume that it's a point dipole located at $r = 0$.

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