AVERAGE FIELD OVER A SPHERE

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The average electric field inside a sphere of radius R due to a point charge q at position $\mathbf{r}$ inside the sphere is the field integrated over the volume of the sphere divided by the sphere's volume:

$$\mathbf{E}_{av}=\frac{1}{4\pi\epsilon_0}\frac{3q}{4\pi R^3}\int\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}d^3\mathbf{r}' \tag{1}$$

where the integral extends over the interior of the sphere. In this case, the source of the field is the charge q at location $\boldsymbol{r}$, and the points at which the field due to this charge is measured are the values of $\boldsymbol{r}'$.

Now suppose we have a uniformly charged sphere with charge density ρ and wish to find the field at point $\mathbf{r}$ due to this charge. In this case, the *sources* of the field range over the entire sphere and are given by the values of $\boldsymbol{r}'$, and the single location at which we measure the resulting field is given by $\boldsymbol{r}$. Thus the roles of $\boldsymbol{r}$and $\boldsymbol{r}'$ are reversed from the previous case.

This time the field at $\mathbf{r}$ due to volume element $d^3\mathbf{r}'$ is $(\mathbf{r}-\mathbf{r}')\,\rho d^3\mathbf{r}'/4\pi\epsilon_0\,|\boldsymbol{r}-\boldsymbol{r}'|^3$ (notice we have interchanged $\boldsymbol{r}$and $\boldsymbol{r}'$ in the formula, although the integral is still over $\boldsymbol{r}'$) so the overall field is

$$\mathbf{E}_{\rho}=\frac{\rho}{4\pi\epsilon_0}\int\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{2}$$

We can take ρ outside the integral as it's assumed to be constant over the volume of the sphere.

The integrals in 1 and 2 are the same (apart from a sign), so the two fields are the same if we set

$$\rho=-\frac{3q}{4\pi R^3}=-\frac{q}{V} \tag{3}$$

where V is the volume of the sphere. That is, ρ is equivalent to smearing the negative of q over the volume of the sphere. Thus to find the average field due to a point charge, we can solve the equivalent problem of finding the field due to a uniform charge distribution, which turns out to be considerably easier.

From Gauss's law, we can work out $\mathbf{E}_\rho$ in 2. Consider first a sphere of radius $r < R$. For such a sphere, the electric field points in the radial direction and has a constant magnitude over the sphere's surface due to symmetry. Gauss's law says that

$$\int_{\mathcal{S}} \mathbf{E}_\rho \cdot d\mathbf{a} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{E}_\rho d^3\boldsymbol{r} \tag{4}$$

where $\mathcal{S}$ is the surface of the sphere and $\mathcal{V}$ is its volume, and

$$\nabla \cdot \boldsymbol{E}_\rho = \frac{\rho}{\epsilon_0} \tag{5}$$

so we have

$$\int_{\mathcal{S}} \mathbf{E}_\rho \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d^3\mathbf{r}' \tag{6}$$

$$4\pi r^2 E_\rho = -\frac{1}{\epsilon_0} \frac{3q}{4\pi R^3} \frac{4\pi r^3}{3} \tag{7}$$

$$E_\rho = -\frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \tag{8}$$

Since $\mathbf{E}_\rho$ points in the radial direction due to symmetry,

$$\mathbf{E}_\rho = -\frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \mathbf{r} \tag{9}$$

The dipole moment of a single point charge q at position $\mathbf{r}$ is $\mathbf{p} = q\mathbf{r}$, so the field can be written as

$$\mathbf{E}_\rho = \mathbf{E}_{av} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3} \tag{10}$$

If $\boldsymbol{r} = 0$ so that the charge is at the centre of the sphere, then $\boldsymbol{E}_{av} = 0$ as we'd expect, since the field over any spherical shell within the sphere points radially in all directions and has the same magnitude over the entire shell.

As we move q (which we'll take to be positive) away from the centre of the sphere, $\boldsymbol{E}_{av}$ becomes non-zero and points opposite to the dipole moment. When the charge reaches the surface of the sphere, so $r = R$, then

$$\boldsymbol{E}_{av} = -\frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{\boldsymbol{r}} \tag{11}$$

The average field becomes the same as the field at $\boldsymbol{r}$ due to a charge of $-q$ placed at the centre of the sphere.

From the superposition principle, we can extend this result so that it applies to any distribution of charge within the sphere.

If $\mathbf{r}$ is outside the sphere, the formula for $\mathbf{E}_{av}$ is the same as before, with the integral still extending over the interior of the sphere. The formula for $\mathbf{E}_{\rho}$ is also the same, but this time if we use Gauss's law and integrate over a spherical surface of radius $r > R$, we get

$$\int_{\mathcal{S}} \mathbf{E}_{\rho} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d^3\mathbf{r}' \tag{12}$$

$$4\pi r^2 E_{\rho} = -\frac{1}{\epsilon_0} \frac{3q}{4\pi R^3} \frac{4\pi R^3}{3} \tag{13}$$

$$E_{\rho} = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \tag{14}$$

$$\boldsymbol{E}_{av} = \mathbf{E}_{\rho} = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \tag{15}$$

The RHS of the second line arises from the fact that all of the charged sphere is now interior to the surface of integration, while on the LHS, we are still integrating over a spherical surface of radius $r > R$. The average field produced by a point charge is thus the same as the field produced by this charge at the centre of the sphere. By superposition we can apply this argument to any distribution of charge external to the sphere.

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