AVERAGE OF PRODUCT OF TWO WAVES

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Post date: 14 Apr 2021.

A common calculation that is required when analyzing any system that varies with a sinusoidal period is a time average over one cycle. For example, a monochromatic plane wave with amplitude A, direction $\mathbf{k}$, frequency ω and phase δ can be written as

$$f = A\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)=\Re\tilde{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{1}$$

$$\tilde{A} = Ae^{i\delta} \tag{2}$$

where $\Re$ indicates the real part.

Now suppose we have two waves with the same direction and frequency, but different amplitudes and phases. Then

$$f = A\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_a) \tag{3}$$

$$g = B\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_b) \tag{4}$$

The average of the product of these waves over a single cycle of time $\frac{2\pi}{\omega}$ is then

$$\langle fg\rangle = \frac{\omega AB}{2\pi}\int_0^{2\pi/\omega}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_a)\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_b)\,dt \tag{5}$$

We can transform this integral by defining

$$\theta \equiv \mathbf{k}\cdot\mathbf{r}-\omega t \tag{6}$$

$$d\theta = -\omega dt \tag{7}$$

$$\langle fg\rangle = \frac{AB}{2\pi}\int_0^{2\pi}\cos(\theta+\delta_a)\cos(\theta+\delta_b)\,d\theta \tag{8}$$

We've used the limits of 0 and 2π since any interval of 2π covers one complete cycle of θ.

The two cosines have the same period and differ only in their phase, so we will get the same result from the integral if we replace them by

$$\cos(\theta+\delta_a)\cos(\theta+\delta_b) \to \cos\theta\cos(\theta+\delta_a-\delta_b) \tag{9}$$
$$= \cos^2\theta\cos(\delta_a-\delta_b) - \cos\theta\sin\theta\sin(\delta_a-\delta_b) \tag{10}$$

We now have

$$\langle fg\rangle = \frac{AB}{2\pi}\cos(\delta_a-\delta_b)\int_0^{2\pi}\cos^2\theta d\theta - \frac{AB}{2\pi}\sin(\delta_a-\delta_b)\int_0^{2\pi}\cos\theta\sin\theta d\theta \tag{11}$$
$$= \frac{1}{2}AB\cos(\delta_a-\delta_b) - 0 \tag{12}$$
$$= \frac{1}{2}AB\cos(\delta_a-\delta_b) \tag{13}$$
$$= \frac{1}{2}\Re(fg^*) = \frac{1}{2}\Re(f^*g) \tag{14}$$

Thus we can get the answer using complex notation without doing any integrals.

This applies to vector products as well, since the components of vector products are just products of scalar functions. For example, the time average of the Poynting vector becomes, when the electric and magnetic fields are written in complex notation:

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\Re\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right) \tag{15}$$

The electromagnetic energy density in the fields has a time average of

$$\langle u_{\text{em}}\rangle = \frac{1}{4}\Re\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \tag{16}$$
$$= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \tag{17}$$
$$= \frac{1}{4\mu_0}\left(\frac{1}{c^2}\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \tag{18}$$

We dropped the $\Re$ in line 2 since the quantity in parentheses is automatically real anyway, and in the last line we used

$$\mu_0\epsilon_0 = \frac{1}{c^2} \tag{19}$$