# BIOT-SAVART LAW

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As mentioned earlier, besides feeling a force from an external magnetic field, an electric current also produces its own magnetic field. The experimentally determined rule for calculating this generated magnetic field is known as the Biot-Savart law. For a steady current (one that doesn't vary with time) in a wire, this law can be written as

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dl' \tag{1}$$

where $\mathbf{r}'$ is a location on the wire and $\mathbf{r}$ is the point at which you want to determine the magnetic field. The integral is a line integral along the path of the current. The constant $\mu_0$ is known as the permeability of free space, and is the magnetic analogue to $\epsilon_0$ in electrostatics. Its value is

$$\mu_0 = 1.25663706\times10^{-6}\text{m kg s}^{-2}\text{Amp}^{-2} \tag{2}$$

Again, this law isn't derived from anything more fundamental; it's a generalization of experiment, although the value of $\mu_0$ is fixed at exactly $4\pi\times10^{-7}\text{m kg s}^{-2}\text{Amp}^{-2}$.

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**Example 1.** Suppose we want to find the field generated by a steady current travelling round a square loop of side length $2R$, at the centre of the square. We can do this by finding the field generated by a single wire segment first.

Because of the cross product in the integrand, the field will be perpendicular to the plane of the square, so if we call this direction the $z$ axis, and set the edge of the square at $x=R$, we can write the integral as (setting $\mathbf{r}=0$ since we're interested in the field at the origin):

$$\begin{aligned}
\mathbf{B}_1(\mathbf{r}) &= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int\frac{|\mathbf{r}-\mathbf{r}'|\sin\theta}{|\mathbf{r}-\mathbf{r}'|^3}dl' && (3)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int_{-R}^{R}\frac{R}{(R^2+y^2)^{3/2}}dy && (4)\\
&= \hat{\mathbf{z}}\frac{\sqrt{2}I\mu_0}{4\pi R} && (5)
\end{aligned}$$

where $\theta$ is the angle between $\mathbf{I}$ and $\mathbf{r}-\mathbf{r}'$, so that $\sin\theta = R/|\mathbf{r}-\mathbf{r}'|$. I used Maple to do the integral, but you can do it by hand by using the substitution $y = R\sin\theta$, so that $dy = R\cos\theta\, d\theta$ and the limits become $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The result, after transforming back to $y$, is that

$$\int\frac{R}{(R^2+y^2)^{3/2}}dy = \frac{y}{R\sqrt{R^2+y^2}} \tag{6}$$

By symmetry, the contribution from all 4 sides is equal, so we get for the total field

$$\mathbf{B} = 4\mathbf{B}_1 = \hat{\mathbf{z}}\frac{\sqrt{2}I\mu_0}{\pi R} \tag{7}$$

**Example 2.** Now suppose we have a current loop consisting of a regular polygon with $n$ sides. In this case, each side subtends an angle of $2\pi/n$, so if we align one side parallel to the $y$ axis at $x = R$, this side will extend from an angle of $-\pi/n$ to $+\pi/n$, and will have a length of $2R\tan\frac{\pi}{n}$. Now the integral for a single side is

$$\begin{aligned}
\mathbf{B}_1(0) &= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int_{-R\tan\frac{\pi}{n}}^{R\tan\frac{\pi}{n}}\frac{R}{(R^2+y^2)^{3/2}}dy && (8)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\frac{2R\tan\frac{\pi}{n}}{R\sqrt{R^2+\left(R\tan\frac{\pi}{n}\right)^2}} && (9)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{2\pi R}\sin\frac{\pi}{n} && (10)
\end{aligned}$$

To get the last line, we used $1+\tan^2 x = \sec^2 x$, so

$$\frac{2R\tan\frac{\pi}{n}}{R\sqrt{R^2+\left(R\tan\frac{\pi}{n}\right)^2}} = \frac{2\tan\frac{\pi}{n}}{R\sec\frac{\pi}{n}} \tag{11}$$

$$= \frac{2}{R}\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}\cos\frac{\pi}{n} \tag{12}$$

$$= \frac{2}{R}\sin\frac{\pi}{n} \tag{13}$$

Each of the $n$ sides will still contribute an equal amount, so the total field is

$$\mathbf{B} = n\mathbf{B}_1 = \hat{\mathbf{z}}\frac{nI\mu_0}{2\pi R}\sin\frac{\pi}{n} \tag{14}$$

This formula reduces to that for a square 7 if $n=4$.

As $n\to\infty$, this formula should give us the field due to a circular loop. In this limit, we can approximate the sine by the first term in its Taylor expansion $\sin\frac{\pi}{n}\approx\frac{\pi}{n}$ so we get

$$\lim_{n\to\infty}\mathbf{B} = \hat{\mathbf{z}}\frac{I\mu_0}{2R} \tag{15}$$

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