COULOMB AND LORENZ GAUGES

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Two common gauges in electrodynamics are the Coulomb gauge and the Lorenz gauge. [Note that Griffiths calls the latter the "Lorentz" gauge, although the gauge was introduced by the Danish physicist Ludvig Lorenz. My thanks to Gerrit Jan van Dijk for pointing this out.] Each gauge amounts to specifying a value for $\nabla\cdot\mathbf{A}$. The Coulomb gauge sets $\nabla\cdot\mathbf{A}=0$, and is the gauge we used when introducing the magnetic vector potential. To see that it's always possible to transform from an arbitrary gauge to the Coulomb gauge, we need to find a function λ such that the transformation

$$\mathbf{A} = \mathbf{A}' + \nabla\lambda \tag{1}$$

$$V = V' - \frac{\partial\lambda}{\partial t} \tag{2}$$

gives $\nabla\cdot\mathbf{A}=0$. To do this, we must have

$$\nabla\cdot\mathbf{A} = 0 = \nabla\cdot\mathbf{A}' + \nabla^2\lambda \tag{3}$$

$$\nabla^2\lambda = -\nabla\cdot\mathbf{A}' \tag{4}$$

The last line is just Poisson's equation and for any "reasonable" original potential $\mathbf{A}'$ it is possible to solve it. In the Coulomb gauge, the potential forms of Maxwell's equations:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{5}$$

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J} \tag{6}$$

reduce to

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{7}$$

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) \tag{8}$$

The scalar potential V is therefore also a solution of Poisson's equation, and once we have found it, we can, in principle, solve the second equation (which is a wave equation with a complicated driving term on the RHS) for the vector potential $\mathbf{A}$.

The Lorenz gauge sets

$$\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t} \tag{9}$$

which transforms 5 and 6 to

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{10}$$

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{11}$$

In terms of the d'Alembertian operator

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \tag{12}$$

we can write these two equations as

$$\Box^2 V = -\frac{\rho}{\epsilon_0} \tag{13}$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{14}$$

so that both potentials now become solutions of the wave equation with a driving term, but now V and $\mathbf{A}$ are decoupled.

Example 1. For the potentials

$$V = 0 \tag{15}$$

$$\mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}\left(ct - |x|\right)^2 \hat{\mathbf{z}} & \text{for } |x| < ct \\ 0 & \text{for } |x| > ct \end{cases} \tag{16}$$

we have

$$\nabla \cdot \mathbf{A} = 0 \tag{17}$$

and because $V = 0$,

$$\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \tag{18}$$

These potentials satisfy the conditions for both the Coulomb and Lorenz gauges.

Example 2. For the potentials of a point charge

$$V(\mathbf{r}, t) = 0 \tag{19}$$

$$\mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{\mathbf{r}} \tag{20}$$

we have

$$\nabla \cdot \mathbf{A} = -\frac{qt}{\epsilon_0} \delta_3(\mathbf{r}) \tag{21}$$

so it uses neither the Coulomb nor Lorenz gauge.

Example 3. For the potentials of an electromagnetic wave

$$V = 0 \tag{22}$$

$$\mathbf{A} = A_0 \sin(kx - \omega t)\, \hat{\mathbf{y}} \tag{23}$$

we again have $\nabla \cdot \mathbf{A} = 0$ so it satisfies both the Coulomb and Lorenz gauges.

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