CURRENTS AND RELATIVITY

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Using the Biot-Savart law and the Lorentz force law, we can see that if we have two parallel wires carrying the same current in the same direction, they will feel a magnetic force of attraction. Using the same logic, we can view the current within a wire (with a non-zero radius) as a number of parallel currents, so it would seem that the current within a wire should be drawn together towards the centre of the wire. The reason this doesn't happen, of course, is because there are also the stationary positive charges within the wire, so if the negative electrons carrying the current clustered towards the centre, the electrical attraction of the positive ions they leave behind will tend to pull them back towards the outside of the wire.

We can make this quantitative by looking at the case of a wire in which the positive charges have a uniform density ρ_+ and are stationary, and the negative charges have a density ρ_- and move relative to the positive ions with a speed v.

The magnetic field due to the current in an infinite wire is

$$B = \frac{\mu_0 I}{2\pi d} \tag{1}$$

where d is the distance from the centre of the wire, and I is the current interior to this distance (current in the cylindrical shell outside d doesn't produce any field within the shell). This current is given by $I = \pi d^2 \rho_- v$, so

$$B = \frac{\mu_0}{2} d\rho_- v \tag{2}$$

The magnetic force on a unit test charge moving with speed v along the wire is then given by the Lorentz force law and is

$$F_B = (1)\, vB = \frac{\mu_0}{2} d\rho_- v^2 \tag{3}$$

The electric field due to an infinite line charge is

$$E = \frac{\lambda}{2\pi\epsilon_0 d} \tag{4}$$

where λ is the net charge density. This result can be obtained by applying Gauss's law

$$\int_{\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q}{\epsilon_0} \tag{5}$$

to a line segment of length 1, and taking the Gaussian surface to be a cylinder of radius d with its axis along the wire. By symmetry, the field points radially away from the wire, so there is no contribution to the Gaussian integral from the ends of the cylinder, and the enclosed charge is $Q = \lambda \times 1 = \lambda$, so we have

$$\int \boldsymbol{E} \cdot d\boldsymbol{a} = E \times 2\pi d \times 1 = 2\pi d E = \frac{\lambda}{\epsilon_0} \tag{6}$$

from which we get 4.

For the linear charge, this means $\lambda = \pi d^2 (\rho_+ + \rho_-)$ (where ρ_- is taken to be negative), so the electric force on a unit charge a distance d from the centre is

$$F_E = (1)\, E = \frac{d(\rho_+ + \rho_-)}{2\epsilon_0} \tag{7}$$

Equating the two forces gives

$$\begin{aligned}
F_B &= F_E && (8)\\
\frac{\mu_0}{2} d\rho_- v^2 &= \frac{d(\rho_+ + \rho_-)}{2\epsilon_0} && (9)\\
\rho_- &= -\frac{\rho_+}{1 - v^2 \mu_0 \epsilon_0} && (10)\\
&= -\frac{\rho_+}{1 - v^2/c^2} && (11)
\end{aligned}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.

Thus it appears that the negative charge density is larger than the positive density, which appears to contradict the supposed neutrality of the wire.

The key to resolving this is to notice the telltale term $\gamma = 1/\sqrt{1 - v^2/c^2}$, that is

$$\rho_- = -\gamma^2 \rho_+ \tag{12}$$

This factor turns up all over the place in special relativity, so let's try analyzing the situation from the two frames of reference: the positive ions and the moving electrons. In the ion frame (the rest frame for the wire), we observe (backed by experimental data) that the wire is neutral, so in that frame, we must have the density of the positive charges equal and opposite to the density of negative charges. If we let ρ_0 be the positive charge density, then in the rest frame

$$\rho_+ = \rho_0 = -\rho_- \tag{13}$$

However, we are actually specifying the densities of the two charge types in two different inertial frames. The positive charges are at rest, but the negative charges are moving with speed v. If these two densities are specified using a unit length in the rest frame, how will they appear in the electron frame?

The unit length of positive charge is at rest in the ion frame, so when we transform to the electron frame, it will undergo Lorentz contraction and appear to have a length of $1/\gamma$. Since the length is smaller, but still contains the same number of charges, the density is higher, so we get

$$\rho'_+ = \gamma\rho_+ = \gamma\rho_0 \tag{14}$$

Now the unit length for negative charge as measured by the ions is moving along at a speed v relative to the ions, so it is already Lorentz contracted. If we shift to the electron frame where this length is at rest, it will appear to *expand* by a factor of γ, which means the density of negative charge appears to *decrease*, giving

$$\rho'_- = \rho_-/\gamma = -\rho_0/\gamma \tag{15}$$

Thus in the electron frame, we get

$$\rho'_+ = -\gamma^2\rho'_- \tag{16}$$

(The reason the positive and negative charges are reversed from what they were in the first derivation above is that traditional electromagnetic theory assumes that current is caused by motion of positive charge, whereas in the relativistic derivation, we assumed that negative charge is moving.)

Thus, according to relativity, a current-carrying wire actually does acquire a net charge density. It turns out that this is true, but a full analysis of how electric fields are calculated in relativity would take us too far afield here.