DIPOLE FORCE AND ENERGY

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In a constant electric field, a perfect dipole feels no force since the force on the positive charge is cancelled by that on the negative charge. However, if the field varies, there could be a net force on the dipole. Over atomic or molecular distances, the field would need to change rapidly in order for there to be any appreciable force. This force is the difference between the forces on the two charges, so we get

$$\mathbf{F} = q\mathbf{E}_+ - q\mathbf{E}_- \tag{1}$$

$$= q\left(\Delta E_x \hat{\mathbf{x}} + \Delta E_y \hat{\mathbf{y}} + \Delta E_z \hat{\mathbf{z}}\right) \tag{2}$$

where

$$\Delta \boldsymbol{E} = \boldsymbol{E}_+ - \boldsymbol{E}_- \tag{3}$$

For an infinitesimal distance, as would be found in an ideal dipole, we can write the change in field as the dot product of the gradient of each component with the infinitesimal displacement $d\mathbf{l}$:

$$\Delta E_x = \nabla E_x \cdot d\mathbf{l} \tag{4}$$

The above relation for the force can then be written as

$$\mathbf{F} = \left(qd\mathbf{l}\cdot\nabla E_x\right)\hat{\mathbf{x}} + \left(qd\mathbf{l}\cdot\nabla E_y\right)\hat{\mathbf{y}} + \left(qd\mathbf{l}\cdot\nabla E_z\right)\hat{\mathbf{z}} \tag{5}$$

$$= q\left(d\mathbf{l}\cdot\nabla\right)\mathbf{E} \tag{6}$$

$$= \left(\mathbf{p}\cdot\nabla\right)\mathbf{E} \tag{7}$$

To work out the energy of a dipole in an electric field, suppose we start with two equal and opposite charges at the same location, and that the electric field is parallel to the z axis. Now we move $+q$ to location $\mathbf{d}/2$ and $-q$ to $-\mathbf{d}/2$. The force on $+q$ is $q\mathbf{E}$ and the work done in moving it is $q\mathbf{E}\cdot\mathbf{d}/2$. Similarly, the work in moving $-q$ is also $-q\mathbf{E}\cdot(-\mathbf{d}/2) = q\mathbf{E}\cdot\mathbf{d}/2$. Thus the total work is $q\mathbf{E}\cdot\mathbf{d} = \mathbf{p}\cdot\mathbf{E}$. If we're working against the force, the potential energy is negative so we get $U = -\mathbf{p}\cdot\mathbf{E}$, which is the standard formula.

However, there are a couple of things about this that are a bit unclear. First, we assumed that $\mathbf{E}$ is constant over the dimensions of the dipole. For an ideal dipole, this is fair enough, since ideal dipoles are essentially point objects. However, suppose we have a field that is constant over all space. In that case, the force on the dipole is zero everywhere, regardless of the orientation or position of the dipole. So we should be able to start the dipole off in its final orientation and move it in from infinity, all with doing zero work. This seems to imply that dipoles have zero energy in a constant field.

But, we also know that rotating a dipole requires work, since the field exerts a torque on the dipole. We've seen that this torque is

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{8}$$

If we start off with $\mathbf{p}$ and $\mathbf{E}$ perpendicular, and then rotate the dipole to some angle θ we have to do work against the torque, which is

$$W = pE\int_{\pi/2}^{\theta} \sin\theta' d\theta' \tag{9}$$

$$= pE\cos\theta \tag{10}$$

$$= \mathbf{p} \cdot \mathbf{E} \tag{11}$$

Thus in one case, we get the energy from separating the two charges against the electric field, and in the second case we get it from rotating the dipole against the field. There's clearly something I'm missing here.

Interaction energy of two dipoles. Anyway, given that the energy has the form given, we can work out the interaction energy of two dipoles. We saw earlier that the field due to a dipole $\mathbf{p}_1$ can be written as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_1\right] \tag{12}$$

The other dipole therefore has an energy within this field of

$$U = -\mathbf{p}_2 \cdot \mathbf{E} \tag{13}$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\right] \tag{14}$$

Force between point charge and dipole. As a final example, suppose we have a dipole and a point charge, with the dipole making an angle θ with the line connecting it with the charge. The force on the dipole due to the charge can be found from the formula 7 for the force. We can take the line connecting the two as the x axis, so the field due to the charge is

$$\mathbf{E}_q = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{15}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\left(x^2+y^2+z^2\right)^{3/2}}\right] \tag{16}$$

The force from 7 is then

$$\mathbf{F}_p = \frac{q}{4\pi\epsilon_0}\mathbf{p}\cdot\nabla\left[\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\left(x^2+y^2+z^2\right)^{3/2}}\right] \tag{17}$$

We get

$$F_{px} = -\frac{q}{4\pi\epsilon_0\left(x^2+y^2+z^2\right)^{5/2}}\left[p_x\left(2x^2-y^2-z^2\right)+3p_yxy+3p_zxz\right] \tag{18}$$

$$= -\frac{q}{4\pi\epsilon_0 r^5}\left[3x\left(p_xx+p_yy+p_zz\right)-p_x\left(x^2+y^2+z^2\right)\right] \tag{19}$$

$$= -\frac{q}{4\pi\epsilon_0 r^5}\left[3x\mathbf{p}\cdot\mathbf{r}-p_xr^2\right] \tag{20}$$

$$F_{py} = -\frac{q}{4\pi\epsilon_0\left(x^2+y^2+z^2\right)^{5/2}}\left[3p_xxy+p_y\left(2y^2-x^2-z^2\right)+3p_zxz\right] \tag{21}$$

$$= -\frac{q}{4\pi\epsilon_0 r^5}\left[3y\mathbf{p}\cdot\mathbf{r}-p_yr^2\right] \tag{22}$$

$$F_{pz} = -\frac{q}{4\pi\epsilon_0\left(x^2+y^2+z^2\right)^{5/2}}\left[3p_xxz+3yzp_y+p_z\left(2z^2-x^2-y^2\right)\right] \tag{23}$$

$$= -\frac{q}{4\pi\epsilon_0 r^5}\left[3z\mathbf{p}\cdot\mathbf{r}-p_zr^2\right] \tag{24}$$

We can combine these to get a single vector expression for **F**:

$$\mathbf{F}_p = \frac{q}{4\pi\epsilon_0 r^3}\left(\mathbf{p}-3\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right) \tag{25}$$

To get the force on the charge, we could just use Newton's third law (equal action and reaction) to say the force is the negative of that on the dipole, but we can also observe this from the formula for the field of a dipole. We get, for a unit vector that points in the opposite direction to that above

$$\mathbf{E}_p = \frac{1}{4\pi\epsilon_0 r^3}\left[3\left(\mathbf{p}\cdot\left(-\hat{\mathbf{r}}\right)\right)\left(-\hat{\mathbf{r}}\right)-\mathbf{p}\right] \tag{26}$$

so the force on the charge is just

$$\mathbf{F}_q = \frac{q}{4\pi\epsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right] \tag{27}$$

$$= -\mathbf{F}_p \tag{28}$$

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