DIPOLES IN THE MULTIPOLE EXPANSION

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Post date: 20 Feb 2021.

We've seen that the potential of a charge distribution can be written as a multipole expansion in terms of Legendre polynomials:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int\rho(\mathbf{r}')P_n(\cos\theta')r'^n d^3\mathbf{r}' \tag{1}$$

where $\mathbf{r}$ is the vector from the origin to the observation point, $\mathbf{r}'$ is the vector from the origin to the point with charge density $\rho(\mathbf{r}')$ and θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

The $n = 0$ term in this expansion is the monopole term, and the potential for that reduces to the equivalent potential for a point charge.

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{Q}{4\pi\epsilon_0 r} \tag{2}$$

The $n = 1$ term is the dipole term, and dominates if the total charge is zero (that is, if there are equal amounts of positive and negative charge).

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\int\rho(\mathbf{r}')P_1(\cos\theta')r'd^3\mathbf{r}' \tag{3}$$

$$= \frac{1}{4\pi\epsilon_0 r^2}\int r'\rho(\mathbf{r}')\cos\theta' d^3\mathbf{r}' \tag{4}$$

Since θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$, we have $\hat{\mathbf{r}}\cdot\mathbf{r}' = r'\cos\theta'$, where $\hat{\mathbf{r}}$ is the unit vector in the $\mathbf{r}$ direction. Since this unit vector is fixed for a given $\mathbf{r}$, we can take it outside the integral and get

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot\int\mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}' \tag{5}$$

The integral is now a property of the charge distribution alone, without any reference to $\mathbf{r}$ (although it does depend on the choice of origin). The integral is a vector and depends on $\mathbf{r}'$, which varies as you move around within the charge distribution. In practice, if the integrand doesn't possess any symmetry that makes it easy to integrate, the only practical way of doing such an integral is to convert to rectangular coordinates, since the unit vectors in the rectangular system are constants and can be taken outside the integral.

The integral is called the *dipole moment* of the charge distribution, and is usually given the symbol **p**:

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}' \tag{6}$$

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot\mathbf{p} \tag{7}$$

For n point charges, the dipole moment can be written as

$$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}'_i \tag{8}$$

Example 1. We have four point charges as follows. Charge $3q$ placed at $(x,y,z) = (0,0,a)$, q at $(0,0,-a)$, $-2q$ at $(0,-a,0)$ and $-2q$ at $(0,a,0)$. We can use 8 to work out the dipole moment.

$$\mathbf{p} = qa(3\hat{\mathbf{z}} - \hat{\mathbf{z}} - 2\hat{\mathbf{y}} + 2\hat{\mathbf{y}}) \tag{9}$$

$$= 2qa\hat{\mathbf{z}} \tag{10}$$

From here, we can get the dipole potential

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\mathbf{p}\cdot\hat{\mathbf{r}} \tag{11}$$

In spherical coordinates, since **p** points along the z axis, we have $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, and

$$V_1(\mathbf{r}) = \frac{2qa\cos\theta}{4\pi\epsilon_0 r^2} \tag{12}$$

Example 2. We have a spherical shell of radius R centred at the origin with a surface charge density of $\sigma = k\cos\theta$ for a constant k (in spherical coordinates), and we want to find the dipole moment.

We can use the symmetry of the situation to note that if we converted the integral to rectangular coordinates, then since the charge density depends only on θ, a charge element at location (x,y,z) will have an equal element at $(-x,-y,z)$, so the components of the dipole moment in the x and y directions will be zero. The component of $\mathbf{r}'$ in the z direction is $R\cos\theta$, so the moment is

$$\mathbf{p} = k\hat{\mathbf{z}} \int_0^{2\pi} \int_0^{\pi} \left(\cos\theta'\right) (R\cos\theta')(R^2 \sin\theta') d\theta' d\phi' \tag{13}$$

$$= \frac{4}{3}\pi k R^3 \hat{\mathbf{z}} \tag{14}$$

The dipole component in the potential is therefore

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\mathbf{p}\cdot\hat{\mathbf{r}} \tag{15}$$

$$= \frac{kR^3}{3\epsilon_0 r^2}\cos\theta \tag{16}$$

Referring back to 1 and using the fact that $\sigma = k\cos\theta = kP_1(\cos\theta)$, we can use the fact that the Legendre polynomials are orthogonal to conclude that this must also be the only non-zero term in the multipole expansion.

Example 3. The classic dipole consists of two equal and opposite charges a distance d apart. Our earlier analysis calculated only the first non-zero term in the expansion, so here we'll look at the problem in more depth.

To make it easier to analyze in spherical coordinates, we place the dipole with both charges on the z axis, and measure the vector **r** from the midpoint of the distance between them. Thus we have a charge $+q$ at $z = d/2$ and $-q$ at $z = -d/2$. We can then define $\mathbf{r}_+$ and $\mathbf{r}_-$ to be the vectors from the two charges to the observation point. With the geometry as given, the angle θ is the angle between the z axis and **r**, so it's the usual spherical angle θ. It is also the angle between **r** and the vector to $+q$, so that $\pi - \theta$ is the angle between **r** and the vector to $-q$.

Using the law of cosines for the sides of a triangle, we get

$$r_+ = \sqrt{r^2 + \frac{d^2}{4} - 2r\frac{d}{2}\cos\theta} \tag{17}$$

$$= r\sqrt{1 - \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \tag{18}$$

$$r_- = \sqrt{r^2 + \frac{d^2}{4} - 2r\frac{d}{2}\cos(\pi-\theta)} \tag{19}$$

$$= r\sqrt{1 + \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \tag{20}$$

The potential at point **r** is therefore

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{21}$$

By substituting for r_+ and r_-, we can then use a Taylor expansion on the variable d/r (which will be small if the charges are close together and we move far away from them) to get the multipole expansion. We get, for the first few terms:

$$V_0 = 0 \tag{22}$$

$$V_1 = \frac{qd}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} \tag{23}$$

$$= \frac{qd}{4\pi\epsilon_0}\frac{P_1(\theta)}{r^2} \tag{24}$$

$$V_2 = 0 \tag{25}$$

$$V_3 = \frac{qd^3}{4\pi\epsilon_0}\frac{5\cos^3\theta - 3\cos\theta}{8r^4} \tag{26}$$

$$= \frac{qd^3}{4\pi\epsilon_0}\frac{P_3(\cos\theta)}{4r^4} \tag{27}$$

$$V_4 = 0 \tag{28}$$

$$V_5 = \frac{qd^5}{4\pi\epsilon_0}\frac{63\cos^5(\theta) - 70\cos^3\theta + 15\cos\theta}{128r^6} \tag{29}$$

$$\frac{qd^5}{4\pi\epsilon_0}\frac{P_5(\cos\theta)}{16r^6} \tag{30}$$

Since r/r_+ is the generating function for the Legendre polynomials, and since $\cos(\pi-\theta) = -\cos\theta$, then r/r_- is the generating function for Legendre polynomials with each term in the sum multiplied by $(-1)^n$. Therefore, since V as given in 21 takes the difference between these two generating functions, all the terms involving even polynomials will cancel out, leaving only the odd terms, as we've seen in the first few explicit terms above.

For large enough r, the dipole term dominates, and we can say that for large distances, the potential of a dipole of two point charges goes as $1/r^2$.

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