DIVERGENCE OF MAGNETIC FIELD - MAGNETIC MONOPOLES

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In electrostatics, Gauss's law states that

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_{0}} \tag{1}$$

that is, that free charge (represented by the charge density ρ) acts as a source or sink for the electric field. Is there an analogous law in magnetostatics?

To find out, we can start with the Biot-Savart law which gives the magnetic field $\mathbf{B}$ in terms of the currents in a volume

$$\mathbf{B}(\mathbf{r})=\frac{\mu_{0}}{4\pi}\int_{V}\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{3}}d^{3}\mathbf{r}' \tag{2}$$

What happens if we take the divergence of this formula? The divergence operator ∇ here takes derivatives with respect to the *unprimed* coordinates only, since the primed coordinates are the variables of integration and don't appear in the final result for $\mathbf{B}$. We get

$$\nabla\cdot\mathbf{B}=\frac{\mu_{0}}{4\pi}\int_{V}\nabla\cdot\left[\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{3}}\right]d^{3}\mathbf{r}' \tag{3}$$

We can use an identity from vector calculus to expand the integrand:

$$\nabla\cdot(\mathbf{A}\times\mathbf{C})=\mathbf{C}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{C}) \tag{4}$$

We can make the correspondences:

$$\mathbf{A}=\mathbf{J}\left(\mathbf{r}'\right) \tag{5}$$

$$\mathbf{C}=\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^{3}} \tag{6}$$

Now $\nabla\times\mathbf{J}(\mathbf{r}')=0$ since $\mathbf{J}$ depends only on primed coordinates. For the other vector, consider the x component:

$$\nabla\times\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)_x = \frac{\partial}{\partial y}\left(\frac{z-z'}{|\mathbf{r}-\mathbf{r}'|^3}\right)-\frac{\partial}{\partial z}\left(\frac{y-y'}{|\mathbf{r}-\mathbf{r}'|^3}\right) \tag{7}$$

$$= -\frac{3}{2|\mathbf{r}-\mathbf{r}'|^5}\left[2\left(y-y'\right)\left(z-z'\right)-2\left(z-z'\right)\left(y-y'\right)\right] \tag{8}$$

$$= 0 \tag{9}$$

where we've used

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{10}$$

The other two components are also zero, so

$$\nabla\times\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right) = 0 \tag{11}$$

and in general

$$\nabla\cdot\mathbf{B} = 0 \tag{12}$$

That is, there are no magnetic sources or sinks, so there is no such thing as 'magnetic charge' or, as they are more commonly known, 'magnetic monopoles'.

Unlike some of the other 'laws' that we've covered, the divergence-free nature of the magnetic field is true not only in magnetostatics, but in more general situations where both the electric and magnetic fields vary with time.

Although magnetic monopoles are excluded from the classical theory of electromagnetism, more modern theories such as string theory do predict their existence. So far, no magnetic monopoles have ever been found.

If magnetic monopoles did exist, then we could make a few changes to the electromagnetic laws we've discussed so far. Basically, these adjustments are creating terms for the magnetic charge analogous to the electric charge. So we would get a divergence for **B**:

$$\nabla\cdot\mathbf{B} = \beta_0\rho_m \tag{13}$$

where β_0 is a constant analogous to ϵ_0 and ρ_m is the magnetic charge density.

Since $\nabla\times\mathbf{B}$ depends on the flow of electric charge, we might expect that $\nabla\times\mathbf{E}$ would now depend on the flow of magnetic charge, so we would get something like

$$\nabla \times \mathbf{E} = \gamma_0 \mathbf{J}_m \tag{14}$$

where γ_0 is another constant and $\mathbf{J}_m$ is magnetic current caused by flowing magnetic monopoles.

The Lorentz force law gives the force resulting from electric charge interacting with a magnetic field, so we might expect an analogous force law resulting from the interaction of a magnetic charge with an electric field, along the lines of

$$\mathbf{F}_m = q_m \mathbf{v} \times \mathbf{E} \tag{15}$$

At this point, we need to examine the units. The electric field was originally defined as the force per unit of *electric* charge, so the units of the RHS here are

$$(\text{magnetic charge})\left(\text{length} \times \text{time}^{-1}\right)\left(\text{mass} \times \text{length} \times \text{time}^{-2} \times \text{electric charge}^{-1}\right) \tag{16}$$

If the unit of magnetic charge is the same as that of electric charge, then the units of the RHS work out to force$\times$velocity so this doesn't work. We could therefore define the units of magnetic charge as $(\text{electric charge})/(\text{velocity})$.

Presumably, a magnetic charge would also be affected by a magnetic field in the same way an electric charge is affected by an electric field, so the total force on a magnetic monopole would be something like

$$\mathbf{F} = q_m \mathbf{B} + q_m \mathbf{v} \times \mathbf{E} \tag{17}$$

However, with the definition of the unit of magnetic charge as given above, the units don't work out correctly for the first term (since $q_e \mathbf{v} \times \mathbf{B}$ has the units of force, $q_m \mathbf{B}$ has units of

$$\underbrace{(\text{electric charge})(\text{velocity})^{-1}}_{[q_m]}\underbrace{(\text{force})(\text{electric charge})^{-1}(\text{velocity})^{-1}}_{[q_e \mathbf{v} \times \mathbf{B}]/[q_e v]=[B]} \tag{18}$$

$$= (\text{force})(\text{velocity})^{-2} \tag{19}$$

where square brackets indicate 'units of'. We would therefore need to multiply the first term by a quantity that has units of velocity^2. If we define some constant velocity as v_0 then we might get something like

$$\mathbf{F} = v_0^2 q_m \mathbf{B} + q_m \mathbf{v} \times \mathbf{E} \tag{20}$$

If we want v_0 to be a universal constant, the fact that it is a velocity suggests that it might be the speed of light c.

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