ELECTRIC DISPLACEMENT

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We've seen that potential due to the polarization density in a dielectric can be represented as bound charges which divide into a volume charge distribution and a surface charge distribution. The relation between these charge distributions and the polarization density $\mathbf{P}$ is

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \tag{1}$$

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{2}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface of the dielectric.

These bound charges can be used to work out the electric field due to the polarization. However in a more general situation, there could be other charges, which we'll call *free charges* not associated with the polarization, and these charges will also generate an electric field. In a 'real' dielectric, there is no real distinction between the volume and surface charge densities, since the volume charge density will not change discontinuously at the surface. We can therefore concentrate on the volume charge density ρ_b in the analysis that follows.

If we add a free charge density ρ_f to the bound charge density, then the total charge density is $\rho = \rho_b + \rho_f$. From Gauss's law we know that $\epsilon_0\nabla\cdot\mathbf{E} = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f$. We can combine the two divergence terms to get

$$\nabla\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho_f \tag{3}$$

This new vector is called the *electric displacement* $\mathbf{D}$:

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \tag{4}$$

The units of $\mathbf{D}$ are those of polarization density, which is dipole moment per unit volume. The dipole moment has units of charge times distance, so the units of $\mathbf{D}$ are charge times distance over volume, or charge per unit area.

Another way of looking at it is in terms of a parallel plate capacitor, initially in a vacuum. If the capacitor has flat plates that carry a charge of σ_f

(positive on one plate, negative on the other), then the electric field between the plates is $E=\sigma_f/\epsilon_0$, pointing from the positive plate to the negative one. Now if we introduce a dielectric between the plates, the field will induce bound charges in the dielectric. In particular, there will be a surface charge $\pm\sigma_b$ on the surfaces of the dielectric in contact with the plates. Thus the effective field between the plates is reduced to $E=\left(\sigma_f-\sigma_b\right)/\epsilon_0$. From the formula above, we can write this as (using 1 with $\boldsymbol{P}\cdot\hat{\boldsymbol{n}}=P$ since the polarization is perpendicular to the plates):

$$\epsilon_0 E+\sigma_b = \sigma_f \quad (5)$$

$$\epsilon_0 E+P = \sigma_f = D \quad (6)$$

Thus the displacement is the density of surface charge required to produce a given field in a capacitor filled with a dielectric. The actual value of P will depend on the material used for the dielectric.

We can integrate the divergence equation and use the divergence theorem to get

$$\int_{\mathcal{V}}\nabla\cdot\mathbf{D}d^3\mathbf{r} = Q_f \quad (7)$$

$$= \int_{\mathcal{S}}\mathbf{D}\cdot d\mathbf{a} \quad (8)$$

where the integral in the first line is taken over a volume and that in the second line is over the surface bounding the volume.

This integral can be used to simplify the calculation of electric field in some situations. For example, if we have a spherical dielectric shell with inner radius a and outer radius b, and this shell has a polarization given by

$$\mathbf{P}=\frac{k}{r}\hat{\mathbf{r}} \quad (9)$$

where k is a constant, we could calculate the electric field by first working out the bound charges and then using Gauss's law. From 1, the bound charges are

$$\sigma_b=\begin{cases}\frac{k}{b} & r=b\\ -\frac{k}{a} & r=a\end{cases} \quad (10)$$

and from 2

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{11}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) \tag{12}$$
$$= -\frac{k}{r^2} \tag{13}$$

Using Gauss's law, in the region $r<a$, there is no enclosed charge, so $E_{r<a}=0$. Inside the shell (that is, for $a<r<b$), we have the contributions from the volume charge and the surface charge on the inner surface of the shell:

$$Q = -\frac{4\pi a^2 k}{a} - 4\pi\int_a^r \frac{k}{r'^2}r'^2 dr' \tag{14}$$
$$= -4\pi kr \tag{15}$$

The field is thus obtained from

$$4\pi r^2\epsilon_0 E = Q \tag{16}$$
$$= -4\pi kr \tag{17}$$
$$E = -\frac{k}{r\epsilon_0} \tag{18}$$

Outside the shell, the net enclosed charge is zero (as we saw in the last post), so again $E=0$.

Using the displacement, however, since there is no free charge anywhere, we have

$$\int_{\mathcal{S}}\mathbf{D}\cdot d\mathbf{a}=0 \tag{19}$$

in all three regions. Inside the inner surface ($r<a$) and outside the outer surface ($r>b$), we know that $\mathbf{P}=0$, so this is equivalent to

$$\int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a}=0 \tag{20}$$

and from the symmetry of the problem, we know that any field would have to be radial, so this gives $E=0$ in these two regions.

Between the surfaces of the shell, again we know that both the field and the polarization density are radial and symmetric, so from 3 and 4 we have $\nabla\cdot\boldsymbol{D}=0$, and the symmetry implies that $\mathbf{D}=0$ as well, which gives

$$\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} \tag{21}$$

$$= -\frac{k}{r\epsilon_0}\hat{\mathbf{r}} \tag{22}$$

It's important to notice that in a more general situation, the equation 19 does not always imply $\mathbf{D}=0$. This is because we need both the divergence and the curl of a vector field to specify it uniquely, and although the curl of $\mathbf{E}$ is always zero in electrostatic problems, there is no corresponding condition on $\mathbf{P}$. In the problem above, we have to invoke the extra conditions imposed by the symmetry of the problem to conclude that $\mathbf{D}=0$; we couldn't conclude that solely from the integral 19.

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