ELECTRIC POTENTIAL - EXAMPLES

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We've seen that the electric field can be expressed as the gradient of a potential function

$$\mathbf{E}=-\nabla V \tag{1}$$

and that the potential can be calculated from a line integral of the field

$$V(\mathbf{r})=-\int_{\mathbf{a}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} \tag{2}$$

where **a** is some arbitrarily chosen reference point. The actual value of **a** doesn't matter, since it always disappears when calculating the field, or when calculating a potential difference. However, if we want to define a potential function, we do need to specify this point.

Calculating V from the field is something of an artificial exercise, since usually it is the field we want, not the potential. However, it's useful to run through a few examples to see how the potential can be calculated in this way.

Example 1. We have a uniformly charged solid sphere with radius R and total charge q. We can find the potential at any point inside or outside the sphere, since we worked out the field as Example 2 in an earlier post, although there we used the charge density ρ rather than the total charge q. For $r<R$:

$$E=\frac{r\rho}{3\epsilon_0} \tag{3}$$

where ρ is the charge density. In terms of q this is

for $r<R$:

$$E=\frac{r\rho}{3\epsilon_0} \tag{4}$$

where ρ is the charge density. In terms of q this is

$$q = \frac{4\pi R^3}{3}\rho \tag{5}$$

$$\rho = \frac{3q}{4\pi R^3} \tag{6}$$

So

$$E = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3} \tag{7}$$

Outside the sphere

$$E = \frac{R^3\rho}{3\epsilon_0 r^2} \tag{8}$$

$$= \frac{q}{4\pi\epsilon_0 r^2} \tag{9}$$

Using infinity as the reference point, the potential outside the sphere is

$$V = -\frac{q}{4\pi\epsilon_0}\int_\infty^r \frac{1}{(r')^2}dr' \tag{10}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{r} \tag{11}$$

Inside, we have

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{R} - \int_R^r \frac{q}{4\pi\epsilon_0}\frac{r'}{R^3}dr' \tag{12}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{2R^3}\left(R^2 - r^2\right)\right) \tag{13}$$

$$= \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \tag{14}$$

Example 2. For an infinitely long charged wire of linear charge density λ we can use the field calculated in Example 3 on the earlier post. The electric field is, for a distance s from the wire

$$E = \frac{\lambda}{2\pi s\epsilon_0} \tag{15}$$

The reference point for the potential calculation in this case is a bit tricky. Suppose we choose some arbitrary distance from the wire, which we'll represent as $s = a$. Then

$$V = -\frac{\lambda}{2\pi\epsilon_0}\int_a^s \frac{1}{s'}ds' \quad (16)$$
$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{s}{a} \quad (17)$$

Choosing either $a=\infty$ or $a=0$ doesn't work, since the logarithm term blows up at both points. However, since the reference point is arbitrary, we might just as well leave it at some finite, non-zero value of a.

Since the potential depends only on the distance s, the gradient in cylindrical coordinates contains only a radial term, and we get

$$\mathbf{E} = -\nabla V \quad (18)$$
$$= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{a}{s}\frac{1}{a}\right]\hat{\mathbf{s}} \quad (19)$$
$$= \frac{\lambda}{2\pi s\epsilon_0}\hat{\mathbf{s}} \quad (20)$$

Note that a cancels out, so its precise value doesn't matter.

Example 3. A hollow spherical shell contains charge density $\rho = k/r^2$ for $a \le r \le b$. We worked out the field in Example 5 in the earlier post. The electric field is $E=0$ for $r<a$; $E=\frac{k(r-a)}{r^2\epsilon_0}$ for $a<r<b$ and $E=\frac{k(b-a)}{r^2\epsilon_0}$ for $r>b$. Using infinity as the reference point, we can get the potential at the centre of the sphere.

$$V = -\int_\infty^b \frac{k(b-a)}{r^2\epsilon_0}dr - \int_b^a \frac{k(r-a)}{r^2\epsilon_0}dr \quad (21)$$

(There is no integral for $r<a$ since $E=0$ there.) The integrals are all fairly simple, and we get

$$V = \frac{k}{\epsilon_0}\ln\frac{b}{a} \quad (22)$$

We can get the potential in the region $a<r<b$ by evaluating the integrals

$$V = -\int_\infty^b \frac{k(b-a)}{r'^2\epsilon_0}dr' - \int_b^r \frac{k(r'-a)}{r'^2\epsilon_0}dr' \quad (23)$$
$$= \frac{k}{\epsilon_0}\left[1-\frac{a}{r}+\ln\frac{b}{r}\right] \quad (24)$$

Finally, the potential outside the sphere ($r>b$) is

$$V = -\int_{\infty}^{r}\frac{k(b-a)}{r'^2\epsilon_0}dr' \quad (25)$$

$$= \frac{k}{\epsilon_0}\frac{b-a}{r} \quad (26)$$

Note that the potentials are all continuous at the various boundaries (always a good idea to check this!).

Example 4. A coaxial cable has a cylindrical inner core of radius a with uniform volume charge density ρ, and an outer cylindrical shell of radius b with a surface charge density that is of opposite sign to the charge on the core. The surface charge density is such that the cable is electrically neutral. We worked out the field in Example 6 in the earlier post. The field inside the inner cylinder is $E=\frac{s\rho}{2\epsilon_0}$ and between the inner cylinder and the outer cylinder it is $E=\frac{a^2\rho}{2s\epsilon_0}$ where s is the distance from the axis. The potential *difference* between the axis and the outer cylinder is then

$$V(b)-V(0) = -\int_0^b \mathbf{E}\cdot d\mathbf{l} \quad (27)$$

$$= -\int_0^a \frac{s\rho}{2\epsilon_0}ds - \int_a^b \frac{a^2\rho}{2s\epsilon_0}ds \quad (28)$$

$$= -\frac{a^2\rho}{4\epsilon_0}\left(2\ln\frac{b}{a}+1\right) \quad (29)$$

Note that in calculating a potential difference, we don't need to use a reference point for the potential.

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