ELECTRIC POTENTIAL

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We've seen that we can calculate the electric field $\mathbf{E}$ due to a continuous charge distribution by working out the integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dq \tag{1}$$

where $\mathbf{r}'$ is the location of infinitesimal charge element dq. More usefully, if we define a charge density $\rho(\mathbf{r}')$, we can write this as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')d^3\mathbf{r}' \tag{2}$$

where $d\mathbf{r}'$ is a volume element, such as $dx\ dy\ dz$ in Cartesian coordinates.

We can now point out a useful mathematical fact about this equation. Suppose we consider a fixed location $\mathbf{r}'$ and define the function

$$f(\mathbf{r}) \equiv \frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{3}$$

$$= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{4}$$

Now suppose we take the gradient of this function. The definition of the gradient is

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \tag{5}$$

If we look at the x component of the gradient, we get

$$\frac{\partial f}{\partial x} = -\frac{x-x'}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} \tag{6}$$

$$= -\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{7}$$

The other two components work out similarly, so we see that

$$\nabla f(\mathbf{r})=-\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{8}$$

so we can rewrite the electric field integral as

$$\mathbf{E}(\mathbf{r})=-\frac{1}{4\pi\epsilon_0}\int\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{9}$$

Since the gradient operates only on components of $\mathbf{r}$ and the integral is over $\mathbf{r}'$, we can take the gradient symbol outside the integral:

$$\mathbf{E}(\mathbf{r})=-\nabla\left[\frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d^3\mathbf{r}'\right] \tag{10}$$

The quantity in square brackets is a scalar function of position, and is known as the *electric potential* $V(\mathbf{r})$:

$$V(\mathbf{r})\equiv\frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d^3\mathbf{r}' \tag{11}$$

$$\mathbf{E}(\mathbf{r})=-\nabla V(\mathbf{r}) \tag{12}$$

A couple of notes about the potential function. First, it must not be confused with potential *energy* since it is not an energy at all. It doesn't even have the right units. Since electric field was defined as force per unit charge, it has the units of newtons per coulomb. Since electric field is the gradient of potential and the gradient has units of inverse distance, the unit of potential is newton-metres per coulomb. Since force times distance is work, which is measured in units of energy (joules), the unit of potential is therefore joules per coulomb. This unit is known as the *volt*.

Second, since it is only the gradient of the potential that is physically significant, we could add any constant to the potential without changing the physics.

The potential formula can be used for discrete charges by using the Dirac delta function to represent the charges. For example, if we had a charge q at the origin, we could write

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int\frac{q}{|\mathbf{r}-\mathbf{r}'|}\delta(\mathbf{r}')d^3\mathbf{r}' \tag{13}$$

$$=\frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{14}$$

Although the delta function is infinite at $\boldsymbol{r}'=0$ and thus seems to represent a point charge of infinite density, it is only the integral over space that has physical significance. Doing the integral removes the infinity due to the delta function, but we're still left with a potential that blows up at $r = 0$. This result is therefore only an approximation to the true physical situation, in which point charges are (as far as we know) not possible, although there is some dispute over whether the electron might be an actual point charge. Given that the electron has non-zero charge, mass and angular momentum (spin), it's hardly conceivable that such an object could have zero volume. In any case, I tend to think of 14 as giving the electric potential of a charge with very small physical dimensions, and only for distances somewhat removed from the charge itself.

For a collection of discrete charges at different locations, we could use a sum of delta functions. That is, if we had n charges q_i at locations $\mathbf{r}'_i$, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\left[\sum_{i=1}^{n}\frac{q_i}{|\mathbf{r}-\mathbf{r}'_i|}\delta(\mathbf{r}'_i)\right]d^3\mathbf{r}' \tag{15}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\mathbf{r}-\mathbf{r}'_i|} \tag{16}$$

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