ELECTROMAGNETIC WAVES IN VACUUM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 1 Apr 2021.

Starting with Maxwell's equations

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \tag{1}$$
$$\nabla\cdot\mathbf{B}=0 \tag{2}$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$
$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

we can now investigate what happens if we have time-varying electric and magnetic fields in vacuum. In that case, there is no charge or current so $\rho=\mathbf{J}=0$ and we get

$$\nabla\cdot\mathbf{E}=0 \tag{5}$$
$$\nabla\cdot\mathbf{B}=0 \tag{6}$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{7}$$
$$\nabla\times\mathbf{B}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{8}$$

We can transform these equations into separate equations for $\mathbf{E}$ and $\mathbf{B}$ by taking the curl of the last two:

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E} \tag{9}$$
$$=-\nabla\times\frac{\partial\mathbf{B}}{\partial t} \tag{10}$$
$$=-\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{11}$$

In the first line, we used a vector identity. In the second line we set the LHS equal to the curl of 7, and in the last line we used 8.

Since $\nabla\cdot\mathbf{E}=0$ in vacuum, we get

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}} \tag{12}$$

A similar calculation for **B** gives us

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \tag{13}$$

$$= \mu_0 \epsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} \tag{14}$$

$$= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{15}$$

so we get

$$\boxed{\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}} \tag{16}$$

The wave equation can be generalized to higher dimensions. In two dimensions, we can consider the force on a patch of membrane held under tension (as in a drum), and the wave variable is the displacement of the membrane from equilibrium. In three dimensions, we can consider the change in some property of a 3-d substance. For example, we can think of the change in density of a fluid such as water as a sound wave passes through it. A proper derivation of the 3-d wave equation would take us a bit far afield here, so we'll just quote the result. For a scalar field (that is, the quantity that 'waves') f the 3-d wave equation is

$$\nabla^2 f = \frac{1}{v^2} \partial_t f \tag{17}$$

where v is the speed of the wave through the substance.

Given the 3-d wave equation, we can see that each component of **E** and **B** satisfies the wave equation, and that the speed of the wave is the same in both cases, namely

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{18}$$

Experimentally, it was found that $v = c$, the speed of light. This result seems to me to be one of the most magical results in physics. It predicts that electric and magnetic fields, once produced, sustain each other and propagate as a wave. Not only that, it suggests (it doesn't really *predict*) that light is itself an electromagnetic wave.

We can derive a few more properties of the electromagnetic wave by applying Maxwell's equations to solutions of the wave equations. Since any solution of the wave equation can be expressed as a sum (or integral) over sinusoidal functions (that's Fourier analysis), we can consider only sinusoidal solutions from now on. Considering waves that consist of only a single frequency ω that travel in the $+z$ direction and have no dependence on x or y, we can write the solutions as

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \tag{19}$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} \tag{20}$$

where the tilde indicates we're using complex notation, and that the physical wave is the real part. The parameters $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are constants under the assumptions we've made here. Such a wave is called *monochromatic* ("one colour") because it contains only one frequency (and hence, for visible light, only one colour) and *plane* because the wave is constant over any plane perpendicular to the direction of propagation.

We can now apply Maxwell's equations to these solutions. First, an observation about the complex notation. For the fields above, the real parts depend on space and time through a term $\cos(kz-\omega t)$ and the imaginary parts through a term $\sin(kz-\omega t)$. Maxwell's equations involve only first derivatives with respect to space and time, and these derivatives will convert all cosines into sines and vice versa. Therefore, if the real part of $\tilde{\mathbf{E}}$ or $\tilde{\mathbf{B}}$ satisfies Maxwell's equations (as it does), then applying the same equations to the imaginary parts just replaces all cosines by sines and thus the imaginary parts must also be solutions. So it's safe to apply Maxwell's equations to the full complex functions $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$.

In a vacuum, both $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$ from which we get

$$\nabla\cdot\tilde{\mathbf{E}} = ik\tilde{E}_{0z}e^{i(kz-\omega t)} = 0 \tag{21}$$

$$\nabla\cdot\tilde{\mathbf{B}} = ik\tilde{B}_{0z}e^{i(kz-\omega t)} = 0 \tag{22}$$

Since this must be true for all z, we must have

$$\tilde{E}_{0z} = \tilde{B}_{0z} = 0 \tag{23}$$

That is, the wave has only x and y components, so it must be a *transverse* wave: a wave that oscillates in a plane perpendicular to the direction of propagation.

We can now apply $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, which gives

$$\nabla\times\tilde{\mathbf{E}} = \left(-ik\tilde{E}_{0y}e^{i(kz-\omega t)}\right)\hat{\mathbf{x}} + \left(ik\tilde{E}_{0x}e^{i(kz-\omega t)}\right)\hat{\mathbf{y}} \tag{24}$$

$$= \left(i\omega\tilde{B}_{0x}e^{i(kz-\omega t)}\right)\hat{\mathbf{x}} + \left(i\omega\tilde{B}_{0y}e^{i(kz-\omega t)}\right)\hat{\mathbf{y}} \tag{25}$$

so

$$\tilde{B}_{0x} = -\frac{k}{\omega}\tilde{E}_{0y} \tag{26}$$

$$\tilde{B}_{0y} = \frac{k}{\omega}\tilde{E}_{0x} \tag{27}$$

which can be written in vector form as

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_0 = \frac{1}{c}\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_0 \tag{28}$$

Therefore, not only are $\mathbf{B}$ and $\mathbf{E}$ transverse, they are also perpendicular to each other.

Now suppose we want a monochromatic, plane wave that travels in some arbitrary direction given by $\hat{\mathbf{k}}$. The value of the wave function at some point $\mathbf{r}$ in 3-d space depends on the projection of $\mathbf{r}$ onto the direction of propagation, since for a plane wave, the wave function depends only on the distance we've moved along the direction of propagation. This projection is $\mathbf{r}\cdot\hat{\mathbf{k}}$. We can therefore define the wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ that points along the propagation direction and replace the kz in the equations above by $\mathbf{k}\cdot\mathbf{r}$, so a monochromatic plane wave travelling in direction $\hat{\mathbf{k}}$ is then

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{29}$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{30}$$

For such a wave, the directions given by $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are fixed, and it's conventional to take the direction of $\tilde{\mathbf{E}}_0$ as the polarization direction $\hat{\mathbf{n}}$. Since $\tilde{\mathbf{B}}$ is perpendicular both to $\tilde{\mathbf{E}}$ and $\mathbf{k}$ its direction is given by $\hat{\mathbf{k}}\times\hat{\mathbf{n}}$ so

$$\tilde{\mathbf{E}} = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \tag{31}$$

$$\tilde{\mathbf{B}} = \tilde{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{k}}\times\hat{\mathbf{n}} \tag{32}$$

$$= \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}} \tag{33}$$

Example 1. Suppose we have a monochromatic plane wave of (real) amplitude E_0, frequency ω and phase angle $\delta = 0$. The wave is travelling in the $-x$ direction and polarized in the z direction. We then have

$$\mathbf{k} = -k\hat{\mathbf{x}} \tag{34}$$
$$\hat{\mathbf{n}} = \hat{\mathbf{z}} \tag{35}$$
$$\mathbf{k}\cdot\mathbf{r} = -kx \tag{36}$$
$$\tilde{E}_0 = E_0 \tag{37}$$
$$\mathbf{E} = E_0 \cos(-kx-\omega t)\,\hat{\mathbf{z}} \tag{38}$$
$$\mathbf{B} = \frac{E_0}{c}\cos(-kx-\omega t)\,\hat{\mathbf{y}} \tag{39}$$

Example 2. Now we have the same wave, except that it is travelling in the direction $[1,1,1]$ and is polarized parallel to the xz plane. This time

$$\mathbf{k} = \frac{k}{\sqrt{3}}[1,1,1] \tag{40}$$

Since $\hat{\mathbf{n}}$ is parallel to the xz plane and $\mathbf{k}$ is perpendicular to $\hat{\mathbf{n}}$, we have

$$\hat{\mathbf{n}} = [n_x, 0, n_y] \tag{41}$$
$$\mathbf{k}\cdot\hat{\mathbf{n}} = 0 \tag{42}$$
$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}[1,0,-1] \tag{43}$$

Thus

$$\mathbf{E} = \frac{E_0}{\sqrt{2}}\cos\left(\frac{k}{\sqrt{3}}(x+y+z)-\omega t\right)[1,0,-1] \tag{44}$$
$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E} \tag{45}$$
$$= \frac{E_0}{\sqrt{6}c}\cos\left(\frac{k}{\sqrt{3}}(x+y+z)-\omega t\right)[-1,2,-1] \tag{46}$$

or, since $\omega = ck$

$$\mathbf{E} = \frac{E_0}{\sqrt{2}}\cos\left(\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right)[1,0,-1] \tag{47}$$
$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E} \tag{48}$$
$$= \frac{E_0}{\sqrt{6}c}\cos\left(\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right)[-1,2,-1] \tag{49}$$

PINGBACKS

Pingback: Electromagnetic waves - energy, momentum and light pressure

Pingback: Electromagnetic waves in matter - normal reflection and transmission

Pingback: Fresnel equations for perpendicular polarization

Pingback: Skin depth of electromagnetic waves in conductors

Pingback: Electromagnetic waves in conductors - phases and amplitudes

Pingback: Wave guides - derivation of the wave equation

Pingback: Average of product of two waves

Pingback: Refraction and dispersion coefficients