ELECTROSTATIC BOUNDARY CONDITIONS

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Problems in electrostatics frequently make use of surface charge distributions, in which charge is imagined as smeared out over a mathematical surface, such as a spherical shell. Crossing such a surface results in a discontinuity in the electric field. Qualitatively, for a plane this is fairly obvious, since if the surface charge distribution consists of, say, positive charge, then the electric field has to point away from the surface on both sides. We can use Gauss's law to work out by how much the electric field is discontinuous.

Suppose we have some surface with a surface charge density of σ. This density may depend on the location, and the surface may be curved, but if we consider a small piece of area of size A, and build a little 'pillbox' that encloses this piece of the surface and extends a tiny distance above and below the surface, then Gauss's law says

$$\oint \mathbf{E}\cdot d\mathbf{a}=\frac{q}{\epsilon_0} \tag{1}$$

where the integral on the left is over the surface of the pillbox and q is the charge enclosed by the pillbox. For a small enough area, to first order σ is a constant across the area, so the total charge enclosed by the pillbox is σA. Similarly, if the area is small enough, $\mathbf{E}$ is constant across the area (although a different constant on each side of the surface), so that $\oint \mathbf{E}\cdot d\mathbf{a}=\left(E_{\perp}^{above}-E_{\perp}^{below}\right)A$. (The minus sign arises since $d\mathbf{a}$ points in opposite directions on the two faces of the pillbox.) We therefore get

$$E_{\perp}^{above}-E_{\perp}^{below}=\frac{\sigma}{\epsilon_0} \tag{2}$$

Thus the perpendicular component of the field has a discontinuity of σ/ϵ_0 as we cross the surface. (The contributions to the surface integral from the sides of the pillbox can be made as small as we like by decreasing the thickness of the pillbox, so that its two faces lie essentially right in the surface itself.)

What about the component of $\mathbf{E}$ that is parallel to the surface? From Stokes's theorem, since $\nabla\times\mathbf{E}=0$ in electrostatics, we know that the line integral of $\mathbf{E}\cdot d\mathbf{l}$ is always zero:

$$\oint\mathbf{E}\cdot d\mathbf{l}=0 \tag{3}$$

Now if we choose a path that is a little rectangle whose plane is perpendicular to the surface, where one side of the rectangle lies above the surface and the opposite side lies below it, then $\oint\mathbf{E}\cdot d\mathbf{l}=E_{\parallel}^{above}l-E_{\parallel}^{below}l$ where l is the length of the side. The minus sign arises from the fact that when we integrate around a rectangle $d\mathbf{l}$ points in opposite directions on opposite sides of the rectangle. (Again, we can make the other two sides of the rectangle (the sides perpendicular to the surface) as small as we like, so there is no contribution from them.) Since the integral is zero, we conclude

$$E_{\parallel}^{above}=E_{\parallel}^{below} \tag{4}$$

That is, the component of $\mathbf{E}$ parallel to the surface is continuous across the surface.

Since the potential difference between two points can be calculated by

$$V(\mathbf{b})-V(\mathbf{a})=-\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\mathbf{l} \tag{5}$$

if we choose the two points to be on opposite sides of the surface, then as the distance between the two points is reduced, eventually the integral will also reduce to zero, since the integrand is the parallel component of $\mathbf{E}$ which we know is continuous. Thus the potential is always continuous across a surface.

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