ELECTROSTATIC PRESSURE

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If an object carries a surface charge distribution, the electric field (due to the charge distribution itself or any other external field) will exert a force on the surface charge. If this charge is constrained to lie on the surface of the object (for example, if the object is a conductor, so that all excess charge lies on the surface), then the electric field will create a pressure (which is force per unit area) by its action on this surface charge.

The question is: what is the force on a surface charge distribution? This is a somewhat tricky question, since we've seen that the component of the electric field that is normal to the surface is discontinuous, with a difference of σ/ϵ_0 from one side of the surface to the other (where σ is the surface charge density).

To answer this question, we can consider some arbitrary surface (not necessarily a conductor) which has surface charge distributed over it. We can also consider a small patch on this surface and examine the fields acting on it. We've seen (Example 1 here) that, for an infinite plane of charge, the field is $\sigma/2\epsilon_0$ on each side of the plane, pointing away from the plane (for positive charge) on both sides. Now if we're considering a small patch of a surface, that's clearly not an infinite plane, but if we also consider the field just above (or below) this patch, then we can say that we're considering the field at a distance from the patch that is very small compared to the dimensions of the patch. In such a case we expect the field to become arbitrarily close to that for an infinite plane.

For example, if we consider the case of a circular disk of charge, the field at location z on the axis of the disk is

$$E = \frac{1}{4\pi\epsilon_0}2\pi\sigma\frac{\left(\sqrt{z^2+R^2}-z\right)}{\sqrt{z^2+R^2}} \tag{1}$$

$$= \frac{\sigma}{2\epsilon_0}\frac{\left(\sqrt{z^2+R^2}-z\right)}{\sqrt{z^2+R^2}} \tag{2}$$

In the limit $z \ll R$ (where R is the diameter of the disk), we see that this expression tends to $\sigma/2\epsilon_0$. Similar limits for other geometries always produce the same result.

So we can split the electric field in the area of the patch into two contributions. The first is due to the patch itself, and is $\sigma/2\epsilon_0$ pointing normal to the surface on both sides, and some other field $\mathbf{E}_1$ which could be anything, depending on the geometry of the surface and other fields in the vicinity. That is, we can say that the fields above and below the patch are

$$\mathbf{E}_{above} = \mathbf{E}_1 + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \quad (3)$$

$$\mathbf{E}_{below} = \mathbf{E}_1 - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \quad (4)$$

We can solve these two equations to find the 'other' field

$$\mathbf{E}_1 = \frac{1}{2}\left(\mathbf{E}_{above} + \mathbf{E}_{below}\right) \quad (5)$$

That is, if we know the field on either side of the patch, we can find the field acting on the patch, and it turns out to be just the average of the field on either side of the patch (that is, on either side of the discontinuity). Furthermore, we can say that $\mathbf{E}_1$ is the only field that acts on the patch, since a charge's field doesn't act on the charge itself. (OK, this argument is a bit dodgy, since we're not considering a point charge, but rather a small patch, so that technically, yes, the field produced by one part of the patch does act on other parts of the same patch. However, we're juggling two limits here: in the first place we're assuming that the size of the patch is small enough that we can consider its field to be almost that due to a point charge so that the field doesn't act back on the patch itself. In the second place, we're considering that the distance above the patch is small relative to the size of the patch so that we say the field due to the patch is $\sigma/2\epsilon_0$. So we're essentially nesting one infinitesimal inside another.)

The argument so far is valid for any surface charge. In the special case of a conductor, we know the fields above and below the surface. Below the surface (that is, inside the conductor) we know that $\mathbf{E}_{below} = 0$, and just above the surface we know that $\mathbf{E}_{above} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$ (the field at the surface of a conductor is always normal to the surface). In this case, the field that acts on the surface charge of a conductor is

$$\mathbf{E}_1 = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \quad (6)$$

and the force per unit area is then just the field times the surface charge density:

$$\mathbf{F} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} \tag{7}$$

Since force per unit area is pressure, we can define the *electrostatic pressure* that the charge exerts on the surface of a conductor as the magnitude of the force per unit area

$$P = \frac{\sigma^2}{2\epsilon_0} \tag{8}$$

$$= \frac{\epsilon_0}{2}E_{above}^2 \tag{9}$$

where E_{above} is the electric field on the outer surface of the conductor:

$$E_{above} = \frac{\sigma}{\epsilon_0} \tag{10}$$

A couple of examples of this pressure:

Example 1. Two large metal plates, each of area A are held a distance d apart. If there is a charge Q on each plate, then the field due to each plate is $E = \sigma/2\epsilon_0$, with $\sigma = Q/A$, pointing away from the plate on each side. Between the plates, $E = 0$, while outside the plates, $E = \sigma/\epsilon_0$ pointing away from the plates. The electrostatic pressure is therefore

$$P = \frac{\epsilon_0}{2}E^2 \tag{11}$$

$$= \frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2} \tag{12}$$

$$= \frac{Q^2}{2\epsilon_0 A^2} \tag{13}$$

Note that the answer does not depend on d, provided that the linear dimensions of the plates are much larger than their separation, so that we can regard them as (almost) infinite planes.

Example 2. A solid spherical conductor of radius R carries a total charge Q. We can find the force of repulsion between two hemispheres of the sphere. From symmetry, the electrostatic pressure is $\frac{\epsilon_0}{2}E^2$ in a radial direction. To find the total force between two hemispheres, we can integrate the z component of the force per unit area (that is, the pressure) over one hemisphere to find the total force on it. The electric field is zero inside the sphere, and $\frac{Q}{4\pi\epsilon_0 R^2}$ just outside the sphere. The z component of the pressure is

$$P_z = P\cos\theta \tag{14}$$
$$= \frac{\epsilon_0}{2}E^2\cos\theta \tag{15}$$
$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2\cos\theta \tag{16}$$

Thus the force acting on one hemisphere is the integral of P_z over θ multiplied by the area $2\pi R^2$ of the hemisphere:

$$F = \frac{\epsilon_0}{2}2\pi R^2\frac{Q^2}{(4\pi\epsilon_0 R^2)^2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \tag{17}$$
$$= \frac{Q^2}{32\pi\epsilon_0 R^2} \tag{18}$$

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