FARADAY'S LAW AND THE BIOT-SAVART LAW

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As an example of the analogy between Faraday's law and Ampère's law, suppose we have a toroidal solenoid with inner radius a, width w and height h, where both w and h are much less than a. The solenoid has N turns in total, and carries a current I that is increasing at the constant rate of $\dot{I}=k$. The magnetic field inside the solenoid is given by Griffiths in his example 5.10 and is

$$B=\frac{\mu_0 NI}{2\pi r} \tag{1}$$

where r is the radial distance. In our case we can approximate r by the constant a since the torus is very thin compared to its radius, so we get

$$B \approx \frac{\mu_0 NI}{2\pi a} \tag{2}$$

$$\Phi \approx \frac{\mu_0 NIhw}{2\pi a} \tag{3}$$

With this approximation, the change in flux is

$$\frac{d\Phi}{dt}=\frac{\mu_0 Nkhw}{2\pi a} \tag{4}$$

Since the magnetostatic case and the Faraday case are mathematically equivalent (provided there is no free charge), we can use the Biot-Savart law to calculate the electric field generated by a steady change in magnetic flux if we replace $\mathbf{J}$ by $-\frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}$, or, in the case of a linear current in the magnetostatic case, we replace I by $-\frac{1}{\mu_0}\frac{\partial\Phi}{\partial t}$.

For example, we can find the electric field at a point on the axis of the torus by using the Biot-Savart law:

$$\mathbf{E}=-\frac{1}{4\pi}\frac{\partial\Phi}{\partial t}\int\frac{d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5}$$

Griffiths works this out in his example 5.6, and the derivation goes like this: the line segment $d\boldsymbol{\ell}'$ is always in the xy plane and is tangent to the torus. The vector $\mathbf{r}-\mathbf{r}'$ points from a point on the torus to a point on the z axis, so it is always perpendicular to $d\boldsymbol{\ell}'$. The cross product $d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')$ makes a constant angle θ with the z axis, and θ is also the angle between $\mathbf{r}-\mathbf{r}'$ and the xy plane. By symmetry, any x and y components of the cross product will cancel out in the integration, so we're left with only the z component, which is $|\mathbf{r}-\mathbf{r}'|\cos\theta d\ell$. Furthermore, the magnitude of $\mathbf{r}-\mathbf{r}'$ is always the same, and is $|\mathbf{r}-\mathbf{r}'|=\sqrt{a^2+z^2}$, and $\cos\theta=a/\sqrt{a^2+z^2}$, so

$$\int\frac{d\boldsymbol{\ell}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}=2\pi a\cos\theta\frac{\sqrt{a^2+z^2}}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}}=\frac{2\pi a^2}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}} \tag{6}$$

and the electric field is then

$$\mathbf{E}=-\frac{\partial\Phi}{\partial t}\frac{a^2}{2(a^2+z^2)^{3/2}}\hat{\mathbf{z}}=-\frac{\mu_0 Nkhw}{4\pi}\frac{a}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}} \tag{7}$$

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