FARADAY'S LAW, AMPÈRE'S LAW AND THE QUASISTATIC APPROXIMATION

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Faraday's lawin differential form is

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{1}$$

There is a similarity to Ampère's law, which says

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{2}$$

Considering only electric fields generated from changing magnetic fields (and not those generated by free charges), we then have $\nabla\cdot\mathbf{E}=0$, since there is no free charge. For magnetic fields, $\nabla\cdot\mathbf{B}=0$ always. Once we specify both the curl and divergence of a vector field, the field is determined uniquely (up to a constant), so Faraday's law is formally equivalent to Ampère's law, except that curl is determined by $-\frac{\partial\mathbf{B}}{\partial t}$ instead of $\mu_0\mathbf{J}$. In particular, we can use the right hand rule to determine the direction of $\mathbf{E}$ if we know $-\frac{\partial\mathbf{B}}{\partial t}$ and we can use Ampèrian loops to calculate $\mathbf{E}$ in those problems where the symmetry makes the calculation easier.

There is one important difference between Faraday's and Ampère's law, however. Ampère's law assumes that the currents $\mathbf{J}$ are steady, that is, that there is no dependence on time. Faraday's law clearly does depend on time, in that an electric field is generated only if the magnetic field is changing. In differential form, this doesn't pose a problem, since both sides of the equation refer to a single point (x,y,z). However, in its original integral form:

$$\oint\mathbf{E}\cdot d\boldsymbol{\ell}=-\int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} \tag{3}$$

the t variable is assumed to be the same at all points in the integrals. If we choose some enormous loop for the integral on the left, then any change in $\mathbf{B}$, even one in some small, remote corner of the area enclosed by the loop, is implicitly assumed to affect $\mathbf{E}$ instantaneously around the entire loop.

This is a problem inherent in all non-relativistic physics. In Newton's gravitational theory, for example, no provision is made for any travel time from one mass to the other; if the sun were to suddenly lose half its mass, say, the effect would be felt at the Earth immediately. In reality, of course, nothing can travel faster than the speed of light, so changes in one part of a system will not be felt at other parts until some signal informing these remote parts of the change has reached them. In the case of electromagnetism, the signal speed happens to be exactly that of light, so when we apply Faraday's law in integral form, we really need to take this into account.

In practice, when we're dealing with finite electrical circuits or situations within an Earth-bound laboratory, the distances are usually so short that we can make the approximation that the travel time is zero. This is known as the *quasistatic approximation*.

quasistatic approximation

Example 1. We have an infinite solenoid with n turns per unit length and radius a, carrying a time-dependent current $I(t)$. The fact that the solenoid is infinite means that the quasistatic approximation could well break down for large distances, but we'll do the calculation anyway and see what we get.

Inside the solenoid, the field is

$$\mathbf{B} = n\mu_0 I(t)\,\hat{\mathbf{z}} \tag{4}$$

so

$$-\frac{\partial \mathbf{B}}{\partial t} = -n\mu_0 \dot{I}(t)\,\hat{\mathbf{z}} \tag{5}$$

Using the analogy between Ampère's and Faraday's laws, since the direction of $-\frac{\partial \mathbf{B}}{\partial t}$ is $-z$, we can apply the right-hand rule to find that the induced electric field is circumferential and in the $-\hat{\boldsymbol{\phi}}$ direction (that is, clockwise, looking towards $-z$). The magnitude of $\mathbf{E}$ is obtained by integrating the field around a circle of radius $r < a$:

$$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = 2\pi r E = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = -\pi r^2 n\mu_0 \dot{I}(t) \tag{6}$$

$$\mathbf{E} = -\frac{1}{2} r n\mu_0 \dot{I}(t)\,\hat{\boldsymbol{\phi}} \tag{7}$$

Outside the solenoid, $\mathbf{B}$ is always zero, so there is no contribution to $\mathbf{E}$ here. A circular integration path at a distance $r > a$ still contains the flux inside the solenoid, so

$$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = 2\pi r E = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = -\pi a^2 n\mu_0 \dot{I}(t) \tag{8}$$

$$\mathbf{E} = -\frac{1}{2}\frac{a^2}{r} n\mu_0 \dot{I}(t)\,\hat{\boldsymbol{\phi}} \tag{9}$$

The quasistatic approximation here implies that the value of $\boldsymbol{E}(t)$ at time t is due to the value of $\dot{I}(t)$ for all locations on the solenoid. That is, the effect of the current changing at some distant point on the solenoid is felt immediately everywhere. A more precise calculation must recognize that the contribution to $\boldsymbol{E}$ at a particular location z_0, say, at time t is due to the effects of changes in the current at various times $\tau(z)$ in the past where $\tau(z)$ is the time taken for the signal to travel from z to z_0. If the signal travels at the speed of light c, then for locations on the z axis

$$\tau(z) = \frac{|z - z_0|}{c} \tag{10}$$

For locations off the z axis, we would need to replace $|z - z_0|$ by the distance from a point on the solenoid to the observation point. We would therefore need to replace 7 and 9 with integrals over past times.

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