FIELD OF A POLARIZED CYLINDER

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As another example of the use of the bound charges representation of a polarized object, we can look at a cylinder which contains a uniform polarization **P** perpendicular to its axis. In this case, $\nabla\cdot\mathbf{P}=0$ everywhere inside the cylinder. If we take the direction of **P** to be $\phi=0$ then $\mathbf{P}\cdot\hat{\mathbf{n}}=P\cos\phi$. We thus must find the potential of an infinite cylinder with a surface charge of

$$\sigma_b=P\cos\phi \tag{1}$$

We've worked out the general solution to Laplace's equation in cylindrical coordinates before, so we can use the results from there. The solution inside the cylinder is

$$V_{\text{in}}=B_{\text{in}}+\sum_{n=1}^{\infty}\left[A_nr^n\sin n\phi+B_nr^n\cos n\phi\right] \tag{2}$$

while outside it is

$$V_{\text{out}}=B_{\text{out}}+\sum_{n=1}^{\infty}\left[\frac{D_n}{r^n}\cos n\phi-\frac{C_n}{r^n}\sin n\phi\right] \tag{3}$$

From the boundary condition requiring the potential to be continuous at the surface of the cylinder we get the relations

$$\begin{aligned}B_{\text{out}} &= B_{\text{in}} &(4)\\ C_n &= -A_nR^{2n} &(5)\\ D_n &= B_nR^{2n} &(6)\end{aligned}$$

where R is the radius of the cylinder.

From the condition on the derivative of the potential at the boundary, we get

$$\sum_{n=1}^{\infty}\left[2nR^{n-1}A_n\right]\sin n\phi+\sum_{n=1}^{\infty}\left[2nR^{n-1}B_n\right]\cos n\phi=\frac{\sigma_b}{\epsilon_0} \tag{7}$$

From 1 we see that all coefficients of the sine terms are zero, as are all coefficients of cosine terms except for $n = 1$. We therefore get

$$B_1 = \frac{P}{2\epsilon_0} \tag{8}$$

$$D_1 = R^2 B_1 \tag{9}$$

$$= \frac{PR^2}{2\epsilon_0} \tag{10}$$

The potential in the two regions is thus

$$V_{\text{in}} = \frac{P}{2\epsilon_0} r\cos\phi \tag{11}$$

$$V_{\text{out}} = \frac{PR^2}{2r\epsilon_0}\cos\phi \tag{12}$$

From this we can get the field by taking the negative gradient. Since $r\cos\phi = x$, the field inside the cylinder is

$$\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}} \tag{13}$$

$$= -\frac{P}{2\epsilon_0}\hat{\mathbf{x}} \tag{14}$$

The interior field is thus uniform, just as the field inside a uniformly polarized sphere is uniform.

Outside the cylinder we need to take the gradient in cylindrical coordinates. We get

$$\mathbf{E}_{\text{out}} = -\frac{\partial V_{\text{out}}}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V_{\text{out}}}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{15}$$

$$= \frac{R^2}{2r^2\epsilon_0}\left[P\cos\phi\hat{\mathbf{r}} + P\sin\phi\hat{\boldsymbol{\phi}}\right] \tag{16}$$

We can write this in terms of the polarization vector. If $\mathbf{P}$ points in the direction $\phi = 0$ then we have

$$\mathbf{P} = P\cos\phi\hat{\mathbf{r}} - P\sin\phi\hat{\boldsymbol{\phi}} \tag{17}$$

where the minus sign on the second term is because $\hat{\boldsymbol{\phi}}$ points in the direction of increasing ϕ, which is clockwise from $\phi = 0$. We have then

$$P\cos\phi\hat{\mathbf{r}} + P\sin\phi\hat{\boldsymbol{\phi}} = 2(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{P} \tag{18}$$

so the field is

$$\mathbf{E}_{\text{out}} = \frac{R^2}{2r^2\epsilon_0}\left[2\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{P}\right] \tag{19}$$