FIELD OF A POLARIZED SPHERE

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We saw before that we can write the potential of a polarized object as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

Note that this is a volume integral over the *primed* coordinates $\mathbf{r}'$, that is, over the location of the volume element containing the polarized material.

Although we can work out the field due to a uniformly polarized sphere using the techniques in the last post, it is also possible to do this using this integral directly. For a uniformly polarized sphere, $\mathbf{P}(\mathbf{r}')$ is constant over the volume of the sphere, so we can take it outside the integral.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\mathbf{P}\cdot\int\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{2}$$

The remaining integral depends only on the position $\mathbf{r}$ of the observation point, and not on the polarization vector. We can therefore take $\mathbf{r}$ to lie on the z axis. The angle between $\mathbf{r}$ and $\mathbf{r}'$ is therefore the polar angle θ. By symmetry, only the z component of the vector in the integral will be non-zero, so we can work out that on its own. This is actually the same problem as calculating the electric field due to a sphere that contains a uniform volume charge density ρ. Since each spherical shell within the sphere behaves as a point charge to all points outside the shell, the field at a point outside the sphere ($z > R$) is just

$$E = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3\rho}{3z^2} \tag{3}$$

At a point inside the sphere, from Gauss's law and the symmetry of the sphere, all shells outside the field point contribute nothing, so we get, for $z < R$:

$$E = \frac{1}{4\pi\epsilon_0}\frac{4\pi z^3\rho}{3z^2} \tag{4}$$

$$= \frac{z\rho}{3\epsilon_0} \tag{5}$$

The field thus increases linearly within the sphere and then falls off as an inverse square outside. The integral in 2 is the same as that used in calculating the field with a density $\rho = 4\pi\epsilon_0$, so we have

$$\int \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = \begin{cases} \pm\frac{4\pi R^3}{3r^2}\hat{\mathbf{z}} & r > R \\ \pm\frac{4\pi r}{3}\hat{\mathbf{z}} & r < R \end{cases} \tag{6}$$

where the plus sign is taken if $\mathbf{r}$ points in the $+z$ direction and the minus sign in the other case.

We can now insert the polarization vector so that it makes an angle θ with the observation vector $\mathbf{r}$ and since $\mathbf{P}\cdot\hat{\mathbf{z}} = P\cos\theta$ we have

$$V(\mathbf{r}) = \begin{cases} \frac{R^3}{3\epsilon_0 r^2}P\cos\theta & r > R \\ \frac{r}{3\epsilon_0}P\cos\theta & r < R \end{cases} \tag{7}$$

Note that the rather surprising result that the electric field inside the sphere is uniform. Since $z = r\cos\theta$:

$$\mathbf{E} = -\nabla V_{\text{in}} \tag{8}$$

$$= -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} \tag{9}$$

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