

FIELDS OF A MOVING POINT CHARGE

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We can use the Liénard-Wiechert potentials for a moving point charge to calculate the fields produced by the charge. The potentials are

$$V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r)) \cdot \mathbf{v})} \quad (1)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{v}}{(c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r)) \cdot \mathbf{v})} \quad (2)$$

where $\mathbf{w}(t_r)$ is the position of the particle at the retarded time t_r and $\mathbf{v} = d\mathbf{w}/dt_r$ is the velocity at the same time.

I'll use the shorthand variable

$$\mathbf{r} \equiv \mathbf{r} - \mathbf{w}(t_r) \quad (3)$$

in what follows. Thus the potentials are written as

$$V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} \quad (4)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} \quad (5)$$

Given the potentials, we can work out the fields using

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (6)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (7)$$

Because of the convoluted dependence of the various quantities on the observer's location \mathbf{r} and time t , these derivatives get quite involved. Griffiths works out ∇V in his section 10.3.2, with the result

$$\nabla V = \frac{qc}{4\pi\epsilon_0 (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) \mathbf{v} - (c^2 - v^2 + \mathbf{r} \cdot \mathbf{a}) \mathbf{r}] \quad (8)$$

so we'll deal with $\frac{\partial \mathbf{A}}{\partial t}$ here.

First, it's useful to work out $\dot{t}_r \equiv dt_r/dt$. The retarded time t_r is defined implicitly by the equation

$$\tau = |\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \quad (9)$$

We start with

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial t} \sqrt{\tau \cdot \tau} = \frac{1}{2\tau} \frac{\partial}{\partial t} (\tau \cdot \tau) \quad (10)$$

$$= \frac{1}{2\tau} \frac{\partial}{\partial t} (r^2 - 2\mathbf{r} \cdot \mathbf{w} + w^2) \quad (11)$$

$$= \frac{1}{2\tau} \left(-2\mathbf{r} \cdot \frac{d\mathbf{w}}{dt_r} \dot{t}_r + 2\mathbf{w} \cdot \frac{d\mathbf{w}}{dt_r} \dot{t}_r \right) \quad (12)$$

$$= \frac{1}{\tau} \dot{t}_r \mathbf{v} \cdot (\mathbf{w} - \mathbf{r}) \quad (13)$$

$$= -\frac{1}{\tau} \dot{t}_r \mathbf{v} \cdot \tau \quad (14)$$

From the RHS of 9 we get

$$c \frac{d}{dt} (t - t_r) = c(1 - \dot{t}_r) \quad (15)$$

Combining these two equations we get

$$c(1 - \dot{t}_r) = -\frac{1}{\tau} \dot{t}_r \mathbf{v} \cdot \tau \quad (16)$$

$$\dot{t}_r = \frac{\tau c}{\tau c - \tau \cdot \mathbf{v}} \quad (17)$$

$$= \frac{\tau c}{\tau \cdot \mathbf{u}} \quad (18)$$

where

$$\mathbf{u} \equiv c\hat{\mathbf{r}} - \mathbf{v} \quad (19)$$

Returning to 5 we now have

$$\frac{4\pi c\epsilon_0}{q} \frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} \frac{1}{(c\mathbf{x} - \tau \cdot \mathbf{v})} - \mathbf{v} \frac{1}{(c\mathbf{x} - \tau \cdot \mathbf{v})^2} \frac{\partial}{\partial t} (c\mathbf{x} - \tau \cdot \mathbf{v}) \quad (20)$$

$$= \frac{\partial \mathbf{v}}{\partial t_r} \dot{t}_r \frac{1}{(c\mathbf{x} - \tau \cdot \mathbf{v})} - \mathbf{v} \frac{1}{(c\mathbf{x} - \tau \cdot \mathbf{v})^2} c \frac{\partial \tau}{\partial t} - \frac{\partial \tau}{\partial t} \cdot \mathbf{v} - \tau \cdot \frac{\partial \mathbf{v}}{\partial t_r} \dot{t}_r \quad (21)$$

The derivative

$$\frac{\partial \mathbf{v}}{\partial t_r} \equiv \mathbf{a}(t_r) \quad (22)$$

is the acceleration of the particle at the retarded time. The other two derivatives are, from 9

$$\frac{\partial \mathbf{r}}{\partial t} = c(1 - \dot{t}_r) \quad (23)$$

$$\frac{\partial \mathbf{r}}{\partial t} = -\frac{d\mathbf{w}}{dt_r} \dot{t}_r = -\mathbf{v} \dot{t}_r \quad (24)$$

Therefore

$$\frac{4\pi\epsilon_0 c}{q} \frac{\partial \mathbf{A}}{\partial t} = \dot{t}_r \left[\frac{\mathbf{a}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \frac{\mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} (-c^2 + v^2 - \mathbf{r} \cdot \mathbf{a}) \right] - \frac{c^2 \mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} \quad (25)$$

$$= \frac{\mathbf{r}c}{\mathbf{r}c - \mathbf{r} \cdot \mathbf{v}} \left[\frac{\mathbf{a}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \frac{\mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} (-c^2 + v^2 - \mathbf{r} \cdot \mathbf{a}) \right] - \frac{c^2 \mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} \quad (26)$$

$$= \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [\mathbf{r}c(\mathbf{a}(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \mathbf{v}(c^2 - v^2 + \mathbf{r} \cdot \mathbf{a})) - c^2 \mathbf{v}(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})] \quad (27)$$

$$= \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [(\mathbf{r}c\mathbf{a} - c^2 \mathbf{v})(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \mathbf{r}c\mathbf{v}(c^2 - v^2 + \mathbf{r} \cdot \mathbf{a})] \quad (28)$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0 (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} \left[\left(\frac{\mathbf{r}\mathbf{a}}{c} - \mathbf{v} \right) (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \frac{\mathbf{r}\mathbf{v}}{c} (c^2 - v^2 + \mathbf{r} \cdot \mathbf{a}) \right] \quad (29)$$

Combining this with 8 and inserting into 6 gives the result quoted in Griffiths as equation 10.65:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q\mathbf{r}}{4\pi\epsilon_0 (\mathbf{r} \cdot \mathbf{u})^3} [(c^2 - v^2) \mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})] \quad (30)$$

For reference, the magnetic field is

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A} \quad (31)$$

$$= \frac{1}{c} \hat{\mathbf{t}} \times \mathbf{E}(\mathbf{r}, t) \quad (32)$$

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