FORCE ON A MAGNETIC DIPOLE

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Although the net force on a local, steady current distribution in a constant, uniform magnetic field is zero, this is not true if the field varies in space. One way of showing this (although it constrains the coordinate axes rather artificially) goes like this.
Suppose we have an infinitesimal square loop carrying a current I. The loop has side length ϵ and lies in the yz plane with one corner at the origin. The magnetic field can vary over the loop, but we assume the variation is small enough that a Taylor expansion of the field provides enough accuracy in the calculations. Starting at the origin, we can get a Taylor expansion along the y axis from $y=0$ to $y=\epsilon$:

$$\mathbf{B}(0,y,0)=\mathbf{B}(0,0,0)+y\partial_y\mathbf{B}(0,0,0)+\mathcal{O}\left(\epsilon^2\right) \tag{1}$$

where the notation $\partial_y\mathbf{B}(0,0,0)$ means $\partial\mathbf{B}/\partial y$ evaluated at the origin. Similarly, the field along the segment lying on the z axis is

$$\mathbf{B}(0,0,z)=\mathbf{B}(0,0,0)+z\partial_z\mathbf{B}(0,0,0)+\mathcal{O}\left(\epsilon^2\right) \tag{2}$$

On the top edge (where $z=\epsilon$), we can use the value of the field on the bottom edge as a starting point for the Taylor expansion, so we have

$$\mathbf{B}(0,y,\epsilon)=\mathbf{B}(0,y,0)+\epsilon\partial_z\mathbf{B}(0,y,0)+\mathcal{O}\left(\epsilon^2\right) \tag{3}$$

$$=\mathbf{B}(0,0,0)+y\partial_y\mathbf{B}(0,0,0)+\epsilon\partial_z\mathbf{B}(0,0,0)+\mathcal{O}\left(\epsilon^2\right) \tag{4}$$

In the last line, we've approximated $\partial_z\mathbf{B}(0,y,0)$ by $\partial_z\mathbf{B}(0,0,0)$ since that's the first term in its Taylor expansion. Higher terms (involving the second and higher derivatives of $\mathbf{B}$ are dropped since they contribute terms of order ϵ^2.
Similarly, the field on the right edge is approximated by

$$\mathbf{B}(0,\epsilon,z)=\mathbf{B}(0,0,z)+\epsilon\partial_y\mathbf{B}(0,0,z)+\mathcal{O}\left(\epsilon^2\right) \tag{5}$$

$$=\mathbf{B}(0,0,0)+z\partial_z\mathbf{B}(0,0,0)+\epsilon\partial_y\mathbf{B}(0,0,0)+\mathcal{O}\left(\epsilon^2\right) \tag{6}$$

With these expressions for $\mathbf{B}$, we can use the Biot-Savart law to integrate around the square and get the net force. The general formula is

$$\mathbf{F}=I\int d\mathbf{l}\times\mathbf{B} \tag{7}$$

Starting with the lower edge, we have $d\mathbf{l}=dy\hat{\mathbf{y}}$ (assuming the current is flowing in the $+y$ direction in this edge). The force on this edge is

$$\mathbf{F}_{z=0}=I\int dy\,(B_z\hat{\mathbf{x}}-B_x\hat{\mathbf{z}}) \tag{8}$$

where the subscripts x and z indicate which component of $\mathbf{B}$ we're using. Plugging in the above approximations, we get

$$\mathbf{F}_{z=0}=I\int_0^{\epsilon} dy\,(B_z+y\partial_y B_z)\,\hat{\mathbf{x}}-I\int_0^{\epsilon} dy\,(B_x+y\partial_y B_x)\,\hat{\mathbf{z}} \tag{9}$$

where all magnetic field terms are evaluated at the origin, and are therefore constants. Doing the integrals, we get

$$\mathbf{F}_{z=0}=I\left(\epsilon B_z+\frac{\epsilon^2}{2}\partial_y B_z\right)\hat{\mathbf{x}}-I\left(\epsilon B_x+\frac{\epsilon^2}{2}\partial_y B_x\right)\hat{\mathbf{z}} \tag{10}$$

Along the right edge, $d\mathbf{l}=dz\hat{\mathbf{z}}$ and we have

$$\mathbf{F}_{y=\epsilon}=I\int dz\,(-B_y\hat{\mathbf{x}}+B_x\hat{\mathbf{y}}) \tag{11}$$

$$=I\left[-\epsilon B_y-\frac{\epsilon^2}{2}\partial_z B_y-\epsilon^2\partial_y B_y\right]\hat{\mathbf{x}}+ \tag{12}$$

$$I\left[\epsilon B_x+\frac{\epsilon^2}{2}\partial_z B_x+\epsilon^2\partial_y B_x\right]\hat{\mathbf{y}} \tag{13}$$

Along the top, $d\mathbf{l}=-dy\hat{\mathbf{y}}$ and

$$\mathbf{F}_{z=\epsilon}=I\int dy\,(-B_z\hat{\mathbf{x}}+B_x\hat{\mathbf{z}}) \tag{14}$$

$$=I\left[-\epsilon B_z-\frac{\epsilon^2}{2}\partial_y B_z-\epsilon^2\partial_z B_z\right]\hat{\mathbf{x}}+ \tag{15}$$

$$I\left[\epsilon B_x+\frac{\epsilon^2}{2}\partial_y B_x+\epsilon^2\partial_z B_x\right]\hat{\mathbf{z}} \tag{16}$$

Finally, along the left edge, $d\mathbf{l}=-dz\hat{\mathbf{z}}$ and

$$\mathbf{F}_{y=0} = I\int dz\,(B_y\hat{\mathbf{x}} - B_x\hat{\mathbf{y}}) \tag{17}$$

$$= I\left(\epsilon B_y + \frac{\epsilon^2}{2}\partial_z B_y\right)\hat{\mathbf{x}} - I\left(\epsilon B_x + \frac{\epsilon^2}{2}\partial_z B_x\right)\hat{\mathbf{y}} \tag{18}$$

Adding up the four components of the force we find

$$\mathbf{F} = I\epsilon^2\left[-(\partial_y B_y + \partial_z B_z)\hat{\mathbf{x}} + \partial_y B_x\hat{\mathbf{y}} + \partial_z B_x\hat{\mathbf{z}}\right] \tag{19}$$

Since $\nabla\cdot\mathbf{B} = 0$, the first term is $\partial_x B_x\hat{\mathbf{x}}$ so we get

$$\mathbf{F} = I\epsilon^2\left[\partial_x B_x\hat{\mathbf{x}} + \partial_y B_x\hat{\mathbf{y}} + \partial_z B_x\hat{\mathbf{z}}\right] \tag{20}$$

$$= I\epsilon^2\nabla B_x \tag{21}$$

Because of the way we've oriented the axes, the magnetic moment of the current loop is

$$\mathbf{m} = I\epsilon^2\hat{\mathbf{x}} \tag{22}$$

so the force can be written as

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \tag{23}$$

Although this derivation used a special case in the way the coordinates were aligned, the formula is actually true in general.