FRESNEL EQUATIONS FOR PERPENDICULAR POLARIZATION

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Post date: 4 Apr 2021.

Continuing our study of electromagnetic waves incident on a surface at an oblique angle we'll now use the boundary conditions to derive the reflection and transmission coefficients. The boundary conditions are

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \tag{1}$$
$$B_1^{\perp} - B_2^{\perp} = 0 \tag{2}$$
$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \tag{3}$$
$$\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0 \tag{4}$$

where the subscript 1 refers to fields in medium 1 ($z < 0$) and 2 refers to medium 2 ($z > 0$). That is

$$\mathbf{E}_1 = \mathbf{E}_I + \mathbf{E}_R \tag{5}$$
$$\mathbf{E}_2 = \mathbf{E}_T \tag{6}$$

and similarly for **B**. The suffix I refers to the incident wave, R to the reflected wave and T to the transmitted wave. As we saw last time, the space-time dependence cancels out of the boundary conditions, and we can replace all fields by their (complex) amplitudes so we get

$$\epsilon_1 \left(\tilde{E}_{0_I}^{\perp} + \tilde{E}_{0_R}^{\perp}\right) = \epsilon_2 \tilde{E}_{0_T}^{\perp} \tag{7}$$
$$\tilde{B}_{0_I}^{\perp} + \tilde{B}_{0_R}^{\perp} = \tilde{B}_{0_T}^{\perp} \tag{8}$$
$$\tilde{\mathbf{E}}_{0_I}^{\parallel} + \tilde{\mathbf{E}}_{0_R}^{\parallel} = \tilde{\mathbf{E}}_{0_T}^{\parallel} \tag{9}$$
$$\frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_{0_I}^{\parallel} + \tilde{\mathbf{B}}_{0_R}^{\parallel}\right) = \frac{1}{\mu_2}\tilde{\mathbf{B}}_{0_T}^{\parallel} \tag{10}$$

There are actually two cases to consider: polarization parallel to the incident plane (that is, **E** in the xz plane in our example) or perpendicular to the incident plane (so **E** is polarized along the y axis). Griffiths does the parallel case in his section 9.3.3 so we'll look at the perpendicular case here. (It's important to be clear about what is perpendicular or parallel to what. In the four boundary conditions above, the $\perp$ and $\|$ symbols mean perpendicular and parallel *to the boundary* (that is, the xy plane) *not* the incident plane. The polarization we're considering is perpendicular *to the incident plane*.)

The incident wave travels along wave vector $\mathbf{k}_I$ at an angle of θ_I to the normal to the xy plane, the reflected wave travels along $\mathbf{k}_R$ also at an angle of θ_I, and the transmitted wave travels along $\mathbf{k}_T$ at angle θ_T such that, according to Snell's law

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \tag{11}$$

Condition 7 tells us nothing since **E** is in the y direction, so has no component perpendicular to the xy plane. To use the other conditions, we need to work out the components of **E** and **B**. We know **E** is in the y direction so that's easy. The direction of **B** is given by $\mathbf{k}\times\mathbf{E}$. Consider $\mathbf{B}_I$. Here $\mathbf{k}_I$ points towards the xy plane (from the left) at an angle θ_I to the normal to this plane. The cross product $\mathbf{k}\times\mathbf{E}$ therefore lies in the xz plane and points to the lower right at an angle $\frac{\pi}{2}-\theta_I$ to the normal, so the components of $\mathbf{B}_I$ are (to keep the notation simple in what follows we'll drop the 0 subscript and tilde for the amplitudes):

$$B_{I_z} = B_I\cos\left(\frac{\pi}{2}-\theta_I\right) = \frac{1}{v_1}E_I\sin\theta_I \tag{12}$$

$$B_{I_x} = -B_I\sin\left(\frac{\pi}{2}-\theta_I\right) = -\frac{1}{v_1}E_I\cos\theta_I \tag{13}$$

B_{I_x} is negative since $\mathbf{B}_I$ points towards negative x and positive z. (Remember that, as a consequence of Faraday's law $\boldsymbol{B} = \frac{1}{v}\boldsymbol{k}\times\boldsymbol{E}$.)

Assuming the reflected wave still has polarization in the $+y$ direction, the direction of $\mathbf{B}_R$ is now $\mathbf{k}_R\times\mathbf{E}_R$ and $\mathbf{k}_R$points away (towards the left) from the xy plane at angle $\theta_R=\theta_I$ so $\mathbf{B}_R$ points to the upper right and has components

$$B_{R_z} = B_R\cos\left(\frac{\pi}{2}-\theta_I\right) = \frac{1}{v_1}E_R\sin\theta_I \tag{14}$$

$$B_{R_x} = B_R\sin\left(\frac{\pi}{2}-\theta_I\right) = \frac{1}{v_1}E_R\cos\theta_I \tag{15}$$

Finally, the transmitted wave has direction $\mathbf{k}_T$ which points away (towards the right) from the xy plane, so $\mathbf{B}_T = \frac{1}{v_2}\mathbf{k}_T\times\mathbf{E}_T$ points to the lower right and has components

$$B_{T_z} = B_T \cos\left(\frac{\pi}{2} - \theta_T\right) = \frac{1}{v_2} E_T \sin\theta_T \tag{16}$$

$$B_{T_x} = -B_T \sin\left(\frac{\pi}{2} - \theta_T\right) = -\frac{1}{v_2} E_T \cos\theta_T \tag{17}$$

We're now ready to apply the boundary conditions. First, we use 8, which applies to the z components of **B** so we have

$$\frac{1}{v_1} E_I \sin\theta_I + \frac{1}{v_1} E_R \sin\theta_I = \frac{1}{v_2} E_T \sin\theta_T \tag{18}$$

$$E_I + E_R = \frac{v_1}{v_2} \frac{\sin\theta_T}{\sin\theta_I} E_T \tag{19}$$

$$= \frac{n_1 v_1}{n_2 v_2} E_T \tag{20}$$

$$= E_T \tag{21}$$

where in the penultimate line, we used the condition for the index of refraction $n_i = \frac{c}{v_i}$ where c is the speed of light in vacuum.

Condition 9 just gives us the same relation, so we don't learn anything new from it. Condition 10 applies to B_x only since **B** has no y component.

$$\frac{1}{\mu_1}\left(-\frac{1}{v_1} E_I \cos\theta_I + \frac{1}{v_1} E_R \cos\theta_I\right) = -\frac{1}{\mu_2 v_2} E_T \cos\theta_T \tag{22}$$

$$E_I - E_R = \frac{\mu_1 v_1}{\mu_2 v_2} E_T \frac{\cos\theta_T}{\cos\theta_I} \tag{23}$$

$$= \alpha\beta E_T \tag{24}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \tag{25}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{26}$$

Solving these two equations gives

$$E_R = \frac{1 - \alpha\beta}{1 + \alpha\beta} E_I \tag{27}$$

$$E_T = \frac{2}{1 + \alpha\beta} E_I \tag{28}$$

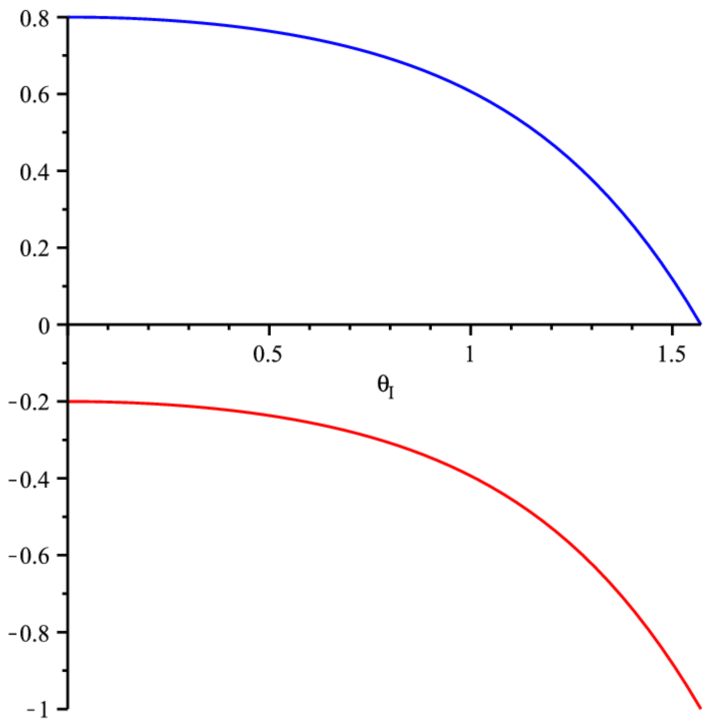

FIGURE 1. Plots of $\frac{E_R}{E_I}$ (red) and $\frac{E_T}{E_I}$ (blue) for $\frac{n_2}{n_1} = 1.5$.

These are Fresnel's equations for perpendicular polarization. For normal incidence, $\theta_I = \theta_T = 0$, $\alpha = 1$ and they reduce to the equations we got in that case. Plots of E_R/E_I (red) and E_T/E_I (blue) for $n_2/n_1 = 1.5$ are in Fig. 1.

The negative values for E_R/E_I indicate that the reflected wave is π out of phase with the incident wave. For $\theta_I = 0$ (normal incidence) 80% of the amplitude is transmitted, dropping to zero when $\theta_I = \pi/2$ (incident wave is parallel to the surface).

The Fresnel equations for parallel polarization (see Griffiths) turn out to be

$$E_R = \frac{\alpha - \beta}{\alpha + \beta} E_I \tag{29}$$

$$E_T = \frac{2}{\alpha + \beta} E_I \tag{30}$$

We can see that if $\alpha = \beta$, $E_R = 0$ and there is no reflected wave. This occurs at ***Brewster's angle*** θ_B, given by

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2} \tag{31}$$

For perpendicular polarization, there is no reflection if we can find an angle θ_B such that $\alpha = 1/\beta$.

$$\alpha^2 = \frac{1-\sin^2\theta_T}{1-\sin^2\theta_I} \tag{32}$$

$$= \frac{1-(n_1/n_2)^2\sin^2\theta_I}{1-\sin^2\theta_I} \tag{33}$$

$$= \frac{1}{\beta^2} \tag{34}$$

$$\sin^2\theta_B = \frac{1-1/\beta^2}{(n_1/n_2)^2-1/\beta^2} \tag{35}$$

$$= \frac{\beta^2-1}{\beta^2(n_1/n_2)^2-1} \tag{36}$$

In practice, the permeabilities of media are approximately equal, so that $\mu_1 \approx \mu_2$ and $\beta \approx n_2/n_1$ from 26. In this case, the expression for $\sin^2\theta_B$ blows up so there is no solution, and thus no Brewster angle for perpendicular polarization (unless $\beta = 1$ which occurs only if $n_1 = n_2$ so there is effectively no boundary).

The reflection and transmission coefficients are

$$R = \frac{\frac{1}{2}\epsilon_1 v_1 E_R^2}{\frac{1}{2}\epsilon_1 v_1 E_I^2} \tag{37}$$

$$= \frac{(1-\alpha\beta)^2}{(1+\alpha\beta)^2} \tag{38}$$

$$T = \frac{\frac{1}{2}\epsilon_2 v_2 E_T^2}{\frac{1}{2}\epsilon_1 v_1 E_I^2} \tag{39}$$

$$= \frac{4\alpha\beta}{(1+\alpha\beta)^2} \tag{40}$$

and it can be seen that $R+T=1$.