

GAUGE TRANSFORMATIONS IN ELECTRODYNAMICS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 3 August 2021.

The potentials V and \mathbf{A} that give rise to a particular configuration of fields \mathbf{E} and \mathbf{B} are not unique. For example, for a point charge at the origin, the usual potentials would be

$$V = \frac{q}{4\pi\epsilon_0 r} \quad (1)$$

$$\mathbf{A} = 0 \quad (2)$$

but as we've seen, the potentials

$$V(\mathbf{r}, t) = 0 \quad (3)$$

$$\mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{\mathbf{r}} \quad (4)$$

give rise to the same fields. The question arises: what is the most general set of potentials that give a particular configuration of fields? That is, what changes can we make to V and \mathbf{A} without changing \mathbf{E} and \mathbf{B} ?

The fields can be calculated from the potentials via the equations

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (5)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (6)$$

We can't multiply V and \mathbf{A} by anything and leave the fields unchanged, but we might be able to add something. Suppose we add a vector function $\boldsymbol{\alpha}(\mathbf{r}, t)$ to \mathbf{A} and a scalar function $\beta(\mathbf{r}, t)$ to V :

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad (7)$$

$$V' = V + \beta \quad (8)$$

To get the same magnetic field, we must have

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (9)$$

$$= \nabla \times \mathbf{A}' \quad (10)$$

$$= \nabla \times (\mathbf{A} + \boldsymbol{\alpha}) \quad (11)$$

$$\nabla \times \boldsymbol{\alpha} = 0 \quad (12)$$

Since its curl is zero, we can write $\boldsymbol{\alpha}$ as the gradient of a scalar field:

$$\boldsymbol{\alpha}(\mathbf{r}, t) = \nabla \lambda'(\mathbf{r}, t) \quad (13)$$

We also have to get the same electric field \mathbf{E} so

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (14)$$

$$= -\nabla(V + \beta) - \frac{\partial(\mathbf{A} + \boldsymbol{\alpha})}{\partial t} \quad (15)$$

$$\nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = 0 \quad (16)$$

Combining these results, we get

$$\nabla \left(\beta + \frac{\partial \lambda'}{\partial t} \right) = 0 \quad (17)$$

The term in parentheses must not depend on position (since its gradient is identically zero everywhere), but it might depend on time. Define this term to be $k(t)$; then

$$\beta = k(t) - \frac{\partial \lambda'}{\partial t} \quad (18)$$

If we define

$$\lambda \equiv \lambda' - \int_0^t k(t') dt' \quad (19)$$

then

$$\beta = -\frac{\partial \lambda}{\partial t} \quad (20)$$

Since λ and λ' differ only by a function of time (and not of space), their gradients are equal, so we can use λ in place of λ' in 13 without changing $\boldsymbol{\alpha}$. Therefore we can change the potentials as follows, without changing the fields they produce:

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \quad (21)$$

$$V' = V - \frac{\partial\lambda}{\partial t} \quad (22)$$

where $\lambda = \lambda(\mathbf{r}, t)$ is an arbitrary scalar field. Such a change to the potentials is called a *gauge transformation*.

Example 1. Earlier, we saw the unusual potentials 3 and 4 for a point charge at the origin. We can transform them using the gauge function

$$\lambda = -\frac{qt}{4\pi\epsilon_0 r} \quad (23)$$

We get

$$\nabla\lambda = \frac{qt}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \quad (24)$$

$$\frac{\partial\lambda}{\partial t} = -\frac{q}{4\pi\epsilon_0 r} \quad (25)$$

so the new potentials are

$$V = \frac{q}{4\pi\epsilon_0 r} \quad (26)$$

$$\mathbf{A} = 0 \quad (27)$$

which is the more usual set of potentials 1 and 2 for a point charge.

PINGBACKS

Pingback: Coulomb and Lorenz gauges