GAUSS'S LAW IN ELECTROSTATICS

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Gauss's law in electrostatics is a relation between the charge contained by a closed surface and the electric field that crosses that surface. The easiest way to see how it works is to begin with a point charge at the origin and a spherical surface centred at the origin. By symmetry the electric field due to the point charge is

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{1}$$

The *flux* of this field through the sphere is defined as the surface integral of the component of the field that is normal to the surface. That is the flux Φ is defined as

$$\Phi=\int\mathbf{E}\cdot d\mathbf{a} \tag{2}$$

where the integral extends over the surface, and $d\mathbf{a}$ is a differential vector whose magnitude is a differential area element and whose direction is normal to the surface at each point.

In the case of a sphere, it is not surprisingly easiest to use spherical coordinates, and in that case

$$d\mathbf{a}=r^2\sin\theta\;d\theta\;d\phi\;\hat{\mathbf{r}} \tag{3}$$

That is, the area element points radially outwards at each point on the sphere.

Combining these results, we see that for a point charge

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} \tag{4}$$
$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot \left(r^2 \sin\theta \; d\theta \; d\phi \; \hat{\mathbf{r}} \right) \tag{5}$$
$$= \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \sin\theta \; d\theta \; d\phi \tag{6}$$
$$= \frac{q}{\epsilon_0} \tag{7}$$

That is, the flux due to a point charge depends only on the magnitude of the charge and not on the radius of the sphere that contains it. This makes intuitive sense, since if we imagine a point charge 'emitting' the electric field, then as long as we provide a surface that wraps up the charge completely, the flux through that surface will be the same regardless of the size of the surface.

In fact, it shouldn't depend on the *shape* of the surface either, so long as that surface completely encloses the charge. To see how this works, suppose we have a charge q enclosed by a closed surface of some arbitrary shape (my diagramming skills aren't up to producing a graphic of this, so try to use your imagination). Draw a vector $\boldsymbol{E}$ representing the electric field from the charge to some element $d\boldsymbol{a}$ on the surface of the shape. This vector makes an angle θ with the unit normal to $d\boldsymbol{a}$, so the component of $\boldsymbol{E}$ normal to the surface at this point is $\boldsymbol{E} \cdot d\boldsymbol{a} = E \; da \; \cos\theta$. If the distance from the charge to this point on the surface is r, then the electric field has a strength of $q/4\pi\epsilon_0 r^2$ at the element $d\boldsymbol{a}$, so that the normal component of the field at this point is

$$E_\perp = \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q\cos\theta}{4\pi\epsilon_0 r^2} \tag{8}$$

The solid angle subtended by the area element $d\boldsymbol{a}$ at the location of the charge is the projection of $d\boldsymbol{a}$ onto a unit sphere centred at the charge. That is, if we take $r = 1$, then the element of solid angle is

$$d\Omega = \hat{\boldsymbol{n}} \cdot d\boldsymbol{a} = da \; \cos\theta \tag{9}$$

If $r \neq 1$, then $d\boldsymbol{a}$ is scaled by $1/r^2$ to give the solid angle, so in this case we have

$$d\Omega = \frac{da}{r^2} \cos\theta \tag{10}$$

Thus from 8 we have

$$\boldsymbol{E}\cdot d\boldsymbol{a}=\frac{q}{4\pi\epsilon_0}d\Omega \tag{11}$$

Thus we see that $\boldsymbol{E}\cdot d\boldsymbol{a}$, when expressed in terms of the solid angle subtended by the area element $d\boldsymbol{a}$, does not depend on r. Thus we can integrate over all solid angle which gives, since $\int_{\text{surface}} d\Omega = 4\pi$:

$$\int_{\text{surface}}\boldsymbol{E}\cdot d\boldsymbol{a}=\int_{\text{surface}}\frac{q}{4\pi\epsilon_0}d\Omega=\frac{q}{\epsilon_0} \tag{12}$$

If the charge is outside the closed surface, then the integral $\int_{\text{surface}}\boldsymbol{E}\cdot d\boldsymbol{a}$ will give some value Ω_+ for the solid angle subtended by the surface at the charge, for the portion of the surface where $\boldsymbol{E}\cdot d\boldsymbol{a}>0$, that is, for the portion of the surface where the field $\boldsymbol{E}$ has a component pointing towards the outside of the surface. However, since the surface is closed, there will be a portion of the surface where $\boldsymbol{E}\cdot d\boldsymbol{a}<0$, that is, where the field $\boldsymbol{E}$ has a component opposite to the surface normal. This portion of the surface subtends a solid angle that is exactly equal to Ω_+, but the contribution to $\int_{\text{surface}}\boldsymbol{E}\cdot d\boldsymbol{a}$ has the opposite sign, so the integral $\int_{\text{outward}}\boldsymbol{E}\cdot d\boldsymbol{a}$ exactly cancels $\int_{\text{inward}}\boldsymbol{E}\cdot d\boldsymbol{a}$, meaning that the integral $\int_{\text{surface}}\boldsymbol{E}\cdot d\boldsymbol{a}=0$ if the charge lies outside the surface.

Notice that this result depends on the fact that the field is an inverse-square force, since it is only in that case that the factor of $\frac{1}{r^2}$ in 8 for the field matches the $\frac{1}{r^2}$ in the expression 10 for the solid angle.

From here, we can generalize the idea to a collection of point charges using the principle of superposition, and get, for a collection of n point charges, all enclosed by the surface:

$$\int\mathbf{E}\cdot d\mathbf{a}=\frac{1}{\epsilon_0}\sum_{i=1}^{n}q_i \tag{13}$$

For a continuous charge distribution, where the charge density is $\rho(\mathbf{r})$, we get

$$\boxed{\int_{\text{surface}}\mathbf{E}\cdot d\mathbf{a}=\frac{1}{\epsilon_0}\int_{\text{volume}}\rho(\mathbf{r})d^3\mathbf{r}} \tag{14}$$

where it is important to note that the integral on the left is over the enclosing surface, while that on the right is over the volume enclosed by that surface. This is the integral form of Gauss's law for electrostatics.

Using the divergence theorem, we can equate the charge density with the divergence of the electric field:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{15}$$

This is the differential form of Gauss's law. Both these forms are very powerful in solving various types of problems since they allow electric fields to be calculated, often without requiring complicated integrals.

References

(1) Griffiths, David J. (2007), *Introduction to Electrodynamics*, 3rd Edition; Pearson Education, Chapter 2.
(2) Jackson, John David (1999), *Classical Electrodynamics*, 3rd Edition; Wiley, Chapter 1.