GREEN'S RECIPROCITY THEOREM

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There is an interesting theorem that relates two separate charge distributions. Suppose we have a charge distribution ρ_1 with its associated potential V_1, and a completely separate charge distribution ρ_2 with potential V_2. These two distributions do not co-exist; they are completely separate situations.

Now consider the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ produced by these two distributions. We can consider the following integral, taken over all space:

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3\mathbf{r} = -\int \nabla V_1 \cdot \mathbf{E}_2 d^3\mathbf{r} \tag{1}$$

Let's consider the first term in the dot product, and use integration by parts:

$$-\int \frac{\partial V_1}{\partial x} E_{2_x} dxdydz = -\int V_1 E_{2_x}\big|_{\text{all }x}\, dydz + \int V_1 \frac{\partial E_{2_x}}{\partial x} dxdydz \tag{2}$$

If we make the usual assumption that the potential V_1 vanishes at infinity then the integrated term (the first term on the RHS) is zero. Doing similar integrals for the other terms in the dot product (integrating with respect to y and then z first for the second and third terms respectively) gives us:

$$-\int \nabla V_1 \cdot \mathbf{E}_2 d^3\mathbf{r} = \int V_1 \nabla \cdot \mathbf{E}_2 d^3\mathbf{r} \tag{3}$$

$$= \frac{1}{\epsilon_0} \int V_1 \rho_2 d^3\mathbf{r} \tag{4}$$

where in the last line we used Gauss's law $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ relating the field to the charge distribution.

We could just as well have done the same calculation interchanging the subscripts 1 and 2, so we get

$$\boxed{\int V_1 \rho_2 d^3\mathbf{r} = \int V_2 \rho_1 d^3\mathbf{r}} \tag{5}$$

which is *Green's reciprocity theorem*.

The result 5 is valid for any two charge distributions, provided that they are not present at the same time. If the distributions are located on conductors, then the potential on each conductor is a constant, so we can take V_i outside the integral, and we get

$$V_1 \int \rho_2 d^3\mathbf{r} = V_2 \int \rho_1 d^3\mathbf{r} \tag{6}$$
$$V_1 Q_2 = V_2 Q_1 \tag{7}$$

For an isolated conductor, the charge, potential and capacitance are related by

$$Q = CV \tag{8}$$

If we have two conductors and place a charge Q_1 on conductor 1 but no extra charge on conductor 2, then the potential of conductor 1 is proportional to Q_1:

$$V_1 = p_{11} Q_1 \tag{9}$$

The potential of conductor 2 is due to charge redistributing itself in response to the field produced by conductor 1. The actual potential V_2 is proportional to Q_1 but is determined by the shape of conductor 2 and its distance from conductor 1. We write this as

$$V_2 = p_{21} Q_1 \tag{10}$$

where p_{21} incorporates the effects of conductor 2's shape and distance from conductor 1.

If the situation is reversed, so that we now place charge Q_2 on conductor 2, but nothing on conductor 1, then

$$V_1 = p_{12} Q_2 \tag{11}$$
$$V_2 = p_{22} Q_2 \tag{12}$$

If we now place charge on *both* conductors, then by the principle of superposition, we can get the potentials by adding the above two situations:

$$V_1 = p_{11} Q_1 + p_{12} Q_2 \tag{13}$$
$$V_2 = p_{21} Q_1 + p_{22} Q_2 \tag{14}$$

Now suppose we take $Q_1 = Q$ and $Q_2 = 0$. Then

$$\begin{aligned} V_1 &= p_{11}Q \\ V_2 &= p_{21}Q \end{aligned} \tag{15}$$

If we reverse the setup, so that $Q_2 = Q$ and $Q_1 = 0$, then we get

$$\begin{aligned} V_1 &= p_{12}Q \\ V_2 &= p_{22}Q \end{aligned} \tag{16}$$

We can use these two setups as the two participants in the reciprocity theorem for conductors in 7. The charge involved in both participants is the same (Q). We'll rewrite 5 with relabelled indices to avoid confusion. We'll call the two configurations a and b so we have

$$\int V_a \rho_b d^3\mathbf{r} = \int V_b \rho_a d^3\mathbf{r} \tag{17}$$

The potential V_a on the LHS consists of the potentials on the two conductors, V_1 and V_2. In each conductor, the corresponding potential is a constant, so we can write the LHS as

$$\int V_a \rho_b d^3\mathbf{r} = V_{a_1} \int \rho_{b_1} d^3\boldsymbol{r} + V_{a_2} \int \rho_{b_2} d^3\boldsymbol{r} \tag{18}$$

Here, ρ_{b_1} is the charge density of conductor 1 in configuration b, and likewise for ρ_{b_2}. The integrals over ρ just give the total charge on each conductor in configuration b**.** In configuration b from 16, so we have $Q_1 = 0$ and $Q_2 = Q$. The potentials V_{a_1} and V_{a_2} are the potentials from configuration a, given by 15. Thus

$$\begin{aligned} \int \rho_{b_1} d^3\boldsymbol{r} &= 0 \\ \int \rho_{b_2} d^3\boldsymbol{r} &= Q \end{aligned} \tag{19}$$

so we have

$$\int V_a \rho_b d^3\mathbf{r} = V_{a_1} \times 0 + V_{a_2} Q \tag{20}$$

$$= p_{21} Q^2 \tag{21}$$

To get the RHS of 17 we use configuration 16 for V_b and

$$\begin{aligned}\int \rho_{a_1} d^3\boldsymbol{r} &= Q \\ \int \rho_{a_2} d^3\boldsymbol{r} &= 0\end{aligned} \tag{22}$$

which gives

$$V_b \rho_a d^3\mathbf{r} = V_{b_1} Q + 0 \tag{23}$$

$$= p_{12} Q^2 \tag{24}$$

Equating 21 and 24 we see that

$$p_{21} = p_{12} \tag{25}$$

In fact, we can generalize all this to a case where we have n conductors. In that case, we have

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \tag{26}$$

and it turns out that the matrix $[p_{ij}]$ is symmetric, as we've shown with the special 2×2 case here. The potential coefficients p_{ij} depend only on the geometry of the setup (shapes and spacings of the conductors) and not on the amounts of charge on the conductors.

Example 1. We have two parallel infinite conducting planes, both of which are grounded. The distance between the plates is d. We place a point charge q between the plates at a distance r from plate 1 (which we take to be the left plate). Find the total charge induced on each plate.

To apply the reciprocity theorem, we need two distinct charge distributions. For the first, we can take the system as described. For the second, we can remove the charge q and also remove the condition that the plates are grounded, so each plate can be at a different potential.

First, consider the distribution as given. Let the potential of the left plate be V_l and of the right plate be V_r. Since the two plates are grounded, we have $V_l = V_r = 0$. Also, since the plates are grounded, the induced charge must cancel out the point charge so there is zero net charge in the system. That is $Q_l + Q_r = -q$.

Now consider the distribution without the point charge q. In this case we can take the potential of the left plate to be $V_l' = 0$ and of the right plate to be $V_r' = V_0$. Note that this assumes the right plate is not grounded. This doesn't matter, since the second distribution can be anything we like. We assume that the plates here have total charges Q'_l and Q'_r, although we'll see we don't need these values anyway.

Since the second distribution contains no point charge, the potential varies linearly between the two plates, so the potential at position r is $V'_r = V'_l + V_0 r/d = V_0 r/d$. Now we're ready to apply the reciprocity theorem. The charge density ρ_2 consists only of the charge on the two plates, since we've removed q. We have, on one side (subscript 1 refers to the configuration with the point charge q; subscript 2 to the configuration without q):

$$\int V_1 \rho_2 d^3\mathbf{r} = V_l Q'_l + V_r Q'_r \tag{27}$$
$$= 0 \tag{28}$$

since $V_l = V_r = 0$.

On the other side, we have, with the point charge given by

$$q\delta(x-r) \tag{29}$$

$$\int V_2 \rho_1 d^3\mathbf{r} = V'_l Q_l + V'_r q + V'_r Q_r \tag{30}$$
$$= V_0 \left(\frac{qr}{d} + Q_r\right) \tag{31}$$

From the theorem, this must be zero, so we get

$$Q_r = -\frac{qr}{d} \tag{32}$$
$$Q_l = -q + \frac{qr}{d} \tag{33}$$
$$= -q\left(1 - \frac{r}{d}\right) \tag{34}$$

Note that the reciprocity theorem in this case allows us to calculate only the *total* charge on each plate; finding the actual surface charge density is a considerably harder problem.

Example 2. We have two concentric spherical conductors of radii a and $b > a$, and a point charge q between them at a location r such that $a < r < b$. Again assuming the spheres are grounded, find the total induced charge on each sphere.

Using similar notation to the last example, we again consider the two distributions to be the original configuration (with the charge q) and a configuration without q. In the first case, since the conductors are grounded, we have

$$V_a = V_b = 0 \tag{35}$$

In the second case, we can take $V_a' = V_0$. In this case, since we don't have the charge q, the system has spherical symmetry, so any charge distributed over the spheres must be uniform, so the potential and the field are the same as if the charge were concentrated at the centre of the spheres. This means that the potential between the spheres has a $1/r$ dependence, so we can write

$$V_a' = V_0 \tag{36}$$

$$V_r' = \frac{a}{r}V_0 \tag{37}$$

$$V_b' = \frac{a}{b}V_0 \tag{38}$$

Applying the reciprocity theorem, we get

$$\int V_1\rho_2 d^3\mathbf{r} = V_a Q_a' + V_b Q_b' \tag{39}$$

$$= 0 \tag{40}$$

On the other side, we have

$$\int V_2\rho_1 d^3\mathbf{r} = V_a' Q_a + V_r' q + V_b' Q_b \tag{41}$$

$$= V_0\left(Q_a + \frac{a}{r}q + \frac{a}{b}Q_b\right) \tag{42}$$

Again, since the two spheres are grounded in the first configuration, we must have $Q_a + Q_b = -q$, so we get

$$\int V_2\rho_1 d^3\mathbf{r} = V_0\left(-q - Q_b + \frac{a}{r}q + \frac{a}{b}Q_b\right) \tag{43}$$

$$= 0 \tag{44}$$

Solving this for Q_b we get

$$Q_b = -q\frac{(r-a)b}{(b-a)r} \tag{45}$$

$$Q_a = -q - Q_b \tag{46}$$

$$= -q\frac{(b-r)a}{(b-a)r} \tag{47}$$